A Practical Guide to AdS/CFT

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Abstract

These lectures, which were delivered at the School of Physics of Technology of the Wuhan University, were intended for an audience familiar with relativistic quantum field theory, but not necessarily non-Euclidean geometry. The goal was to bring the audience as fast as possible to a working knowledge in the field of AdS/CFT.

The first part deals in a very elementary way with those aspects of non-Euclidean geometry which are necessary to perform calculations in AdS/CFT. In the second part dealing with applications, the lectures follow very much the approach of Brodsky and Téramond. The relation between the variable of the 5th dimension with the impact variable, in light cone dynamics, elaborated in this approach, seems to me one of the most interesting results of AdS/CFT.

I have treated technical questions in detail, since I think a certain knowledge of the techniques applied is essential to gain insight into the underlying physics, and that active work on a field, even in petty problems, is the most efficient way to a deeper understanding

It is a pleasure to thank the University of Wuhan, and especially Prof. Liu, Jue Ping, for the warm hospitality extended to me and the listeners of these lectures for their vivid interest and collaboration.

Chapter 1

Generalities

1.1 Introduction

1.1.1 Old string theory in strong interactions

This introductory section is not a practical guide, it only shall give you an idea of the underlying concepts and also to explain some of the highbrow names which occur even in the phenomenological literature.

Theory of strong interactions was a difficult job for theoretical phycisists from the beginning. There were several major steps forward in understanding strong interactions, but most of them turned out to be incomplete. Even today, where we believe to have in **QCD** the final theory of strong interactions we have very little understanding of its nonperturbative sector. We are bound there to rely either on numerical calculations on the lattice or to use models. The so called AdS/CFT theory, which relates nonperturbative QCD with string theory is an attempt to get an **analytical first order approach to nonperturbative effects**.

The first big success in the history of strong interactions was the prediction of the π-Meson (Yukawa 1935), which was indeed detected in 1947. But it turned out that pion field theory could explain some features of strong interactions, but by far not all. Since the coupling of the π-meson to the nucleons is large, the then newly developed renormalized perturbation theory was of little use. The discovery of many "elementary particles" (ρ-Meson, Δ-resonance, Ω^-) etc. led to two opposite strategies:

- Search for elementary **constituents** of the pion, proton etc.
- The so called boot-strap method, an approach based on **duality**.

QCD was a development of the quark model, which is based on the first approach searching for more elementary constituents of the strongly interacting particles (hadrons); it goes back to Fermi and Yang (1949). I do not need to elaborate on QCD here.

The basis of the second approach was **duality**: In this approach one assumed that there were no ultimate fundamental quantum fields which constitute matter, but one had rather a holistic approach to the phenomena. An important concept was that of duality: The sum over all the resonances in the s-channel yields the same result as the sum over the resonaces in the t-channel, this is indicated in figure 1.1

A nice model for scattering amplitudes which show this dual behaviour is the **Veneziano model** $V(s,t)$ (Veneziano 1968). It consists of a sum of expressions like

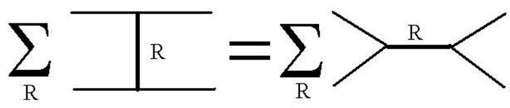

Figure 1.1: Graphical illustration of duality

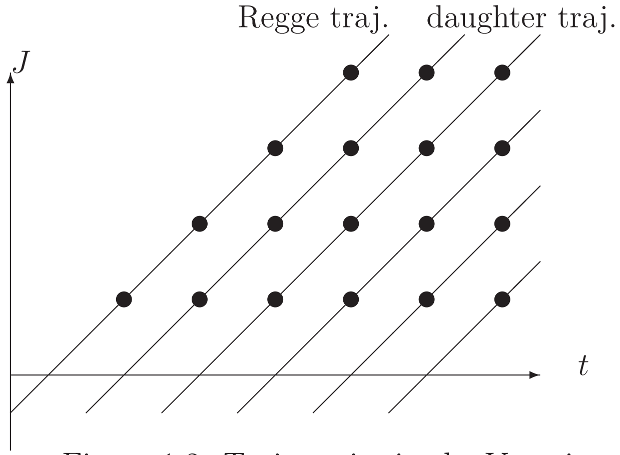

Figure 1.2: Trajectories in the Veneziano model

$$V(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{1.1}$$

with the linear function $\alpha(x) = -\alpha_0 - \alpha' x$. $\Gamma(z)$ is the Euler Gamma function which for integer values is the faculty, $\Gamma(z+1) = z!$. From the properties of the Γ function follows: For large values of s and values of $t < 0$, $V(s,t)$ shows Regge behaviour:

$$V(s,t) \sim s^{\alpha(t)}$$

but it has also the resonance poles for for integer values of $\alpha(s)$ or $\alpha(t)$. These poles lie on straight lines, the highest one is called the Regge trajectory, the lower ones are called daughter trajectories.

It was soon realized, that the Veneziano model corresponds to a string theory, where the rotation of the string gives the resonances along the Regge trajectories and the vibrational modes yield the daughter trajectories, see figure 1.3

In this approach the hadrons are not pointlike objects nor are they composed of pointlike objects (elementary quantum fields), but they are inherently extended objects: strings.

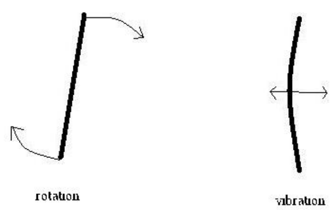

Figure 1.3: Rotational and vibrational modes of a string

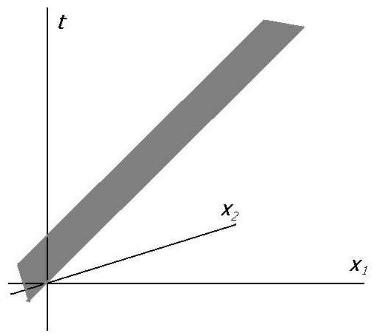

Figure 1.4: The world surface of a string in the $(t, x^{(1)})$ plane, whose ends are parametrized by $\tau = x^4 = ct \quad \text{and} \quad \vec{x}(0,\tau) = (\tau,0,0), \quad \vec{x}(1,\tau) = (1+\tau,0,0)$

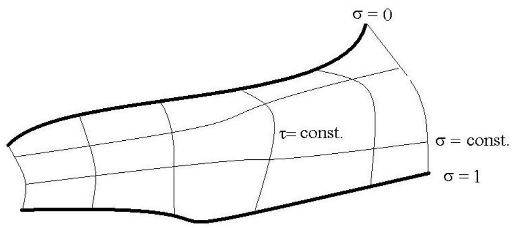

Figure 1.5: Parametrization of a string

A string in 4-dimensional Minkowski space can be parametrized by two parameters, σ, τ:

$$x^\mu(\sigma, \tau)$$

where $x^\mu(0,\tau)$ and $x^\mu(1,\tau)$ are the endpoints of the string.

If we have for instance:

$$\tau = x^4 = ct \quad \text{and} \quad \vec{x}(0,\tau) = (\tau,0,0), \quad \vec{x}(1,\tau) = (1+\tau,0,0),$$

the string is moving with velocity of light along the x^1 axis in the (x^2, x^3) plane, see figure 1.4.

In figure 1.5 a more general parametrization is shown with lines of constant σ and τ indicated on its world surface.

A result for the classical relativistic string is that the angular momentum is proportional to the four momentum squared, that is the (relativistic) mass of the string, $J \sim m^2$ which is just the Regge behaviour. The Veneziano model correponds to a classical string theory, quantum corrections are shown in figure 1.6

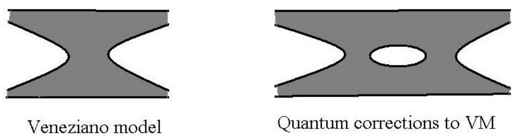

Figure 1.6: Veneziano model and quantum corrections in string theory

Big hopes were put in the Veneziano model and its development, but soon it turned out that it was not the most adequate theory for strong interactions. Beyond internal difficulties one reason was that Quantum Chromodynamics (**QCD**) came out as a strong competitor and now this field theory is generally considered as the correct theory of strong interactions.

String theory however developed in a completely different direction and it is nowadays considered as the best candidate for a quantum theory of gravitation. But it is interesting to note, that string theory in strong interaction physics was never completely dead. The reason is that many aspects of non-perturbative QCD seem to indicate that hadrons have indeed stringlike features. The most poular model for confinement, the t'Hooft-Mandelstam model (see figure 1.7) is based on the assumption that that the colour-electric force lines are compressed (by monople condensation) into a flux tube which behaves in some respect indeed like a string.

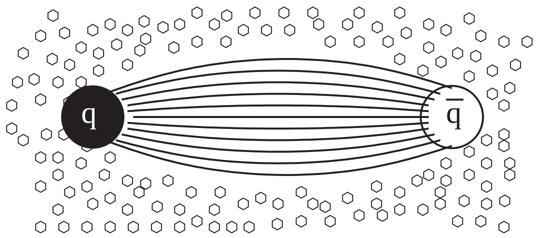

Figure 1.7: Formation of a colourelectric flux tube

Also the particular role of quarks as confined particles shows some analogy with a string picture: If you split a hadron, you do not obtain quarks, but again hadrons. In a similar way, if you cut a string you do not obtain two ends, but two strings again.

The present renewed interest of phenomenologically oriented physicists in this field goes back to conjecture made by Maldacena , Gubser Klebanov and Polyakov, and Witten in 1998[1, 2, 3] [1] It states that a certain string theory is equivalent to a certain Yang-Mills theory. Many people tried to bring this mathematically high-brow theory down to earth and try to learn from string theory some aspects of nonperturbative QCD.

Before we come to formulae, I shall try to tell you in words the principal ideas behind the approach which we shall follow during the rest of the lectures.

1.1.2 AdS/CFT

As mentioned above, there are good reasons to believe, that a certain superstring (Type II B) theory in ten dimensions is dual to a highly supersymmetric (N=4) gauge theory (Maldacena conjecture). Duality means here that the strong coupling regime of one theory reflects the weak coupling regime of the other one. Therefore the classical domain (without loops) of one theory allows **nonperturbative** (strong coupling) calculations in the other theory. This is of course just what we want for QCD.

The relation between the two very different theories comes over the so called D-branes. A D-brane is a hypersurface on which open strings end. Since energy and momentum flows from the string to the D-branes they are also dynamical objects. The D stands by the way for Dirichlet, since the Dirichlet boundary conditions on the D-brane are essential for string dynamics. In the mentioned

[1]for some reviews for non-specialists see e.g. [4, 5] and the very short article [6]

case the D3 branes have 3 space and one time direction and they are boundaries of a 5 dimensional space with maximal symmetry (a so called ant-de Sitter space, we come to this back in detail). In figure 1.8 two D1-branes (1 space, 1 time dimension) are shown, at which an open string ends (picture at a fixed time).

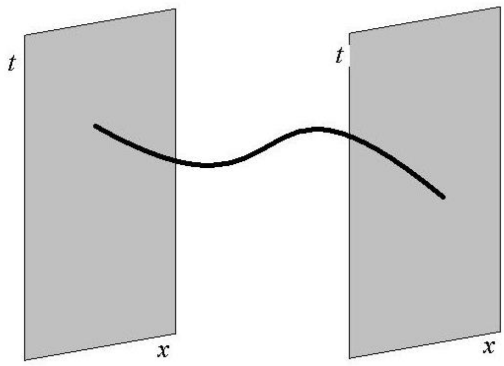

Figure 1.8: Two D1 branes in three-dimensional space

The situation to which the Maldacena conjecture applies is however of no direct use, since the supersymmetric gauge theory of the conjecture is very different from our QCD. The quarks for instance are in this theory necessarily in the same representation as the gauge fields, for QCD this would mean that there are eight coloured quarks. Furthermore this supersymmetric Yang-Mills theory has no intrinsic scale like the QCD scale $\Lambda_{\rm QCD}$. This means that the coupling constant is not running. But the close relation between a superstring (or better supergravity) theory and a Yang Mills theory in one special case and the great need for nonperturbative methods have motivated many physicists to try to obtain nonperturbative results in actual QCD from classical calculations in a higher dimensional space. There are at the moment two principal approaches:

- The top-down approach: One looks for a superstring theory which has as limit on a D3 brane realistic QCD or at least a similar theory. This approach is very difficult and has to my knowledge not yet led to phenomenologically useful results
- The bottom-up approach: One starts with QCD and tries to construct at least an approximate string theory which one can solve and obtain nonperturbative results for QCD

Needless to say that we follow here the bottom-up approach.

Before we come to a more complete mathematical description, I shall outline pictorially the situation. We first go back to the Maldacena conjecture, not to come to a top-down approach, but to learn at least some constraints for the bottom-up construction.

In the Maldacena case the D3 branes at which the strings can end and move, are boundaries of a 5 dimensional space with maximal symmetry, a so called anti-de-Sitter space. Since I cannot draw a picture in 10 dimensions, I try to illustrate it with three dimensions.

If we embed into our three dimensional space a rotational hyperboloid, than this hyperboloid is a space of maximal symmetry (Another example is a sphere, where the symmetry is more apparent; the hyperboloid is an anti-de-Sitter space, the sphere is a de-Sitter space) This is shown in figure 1.9. Values with constant r of the hyperboloid are one-dimensional spaces which represents the D-brane (in the figure it is actually a D0 brane).

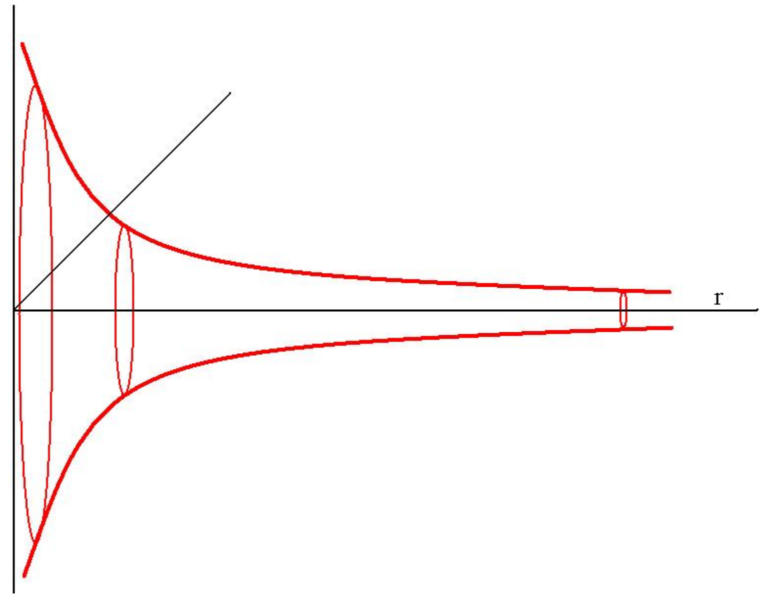

Figure 1.9: A two dimensional anti-de-Sitter space is embedded into a three dimensional space

The maximal symmetry of the 5-dimansional AdS space involves scale invariance. The metric in this space does not change, if we multiply the 4 space-time coordinates x by α and divide the 5th coordinate (r in figure 1.9) by α. If we decrease r by dividing through a large number it correponds to an increase of the distance in ordinary space time. Hence if we can study the r-behaviour in the 5-dimensional AdS it may tell us something about the large distance behaviour in QCD.

The procedure we adopt will be the following: We construct operators in AdS_5 which correspond to **local** QCD operators, e.g. a vector field $\bar{\psi}\Gamma_\mu\psi$ and study the behaviour of this operator in the 5 dimensional space (the so called bulk), especially the r behaviour, and hope to get information on the large-distance, that is nonperturbative, behaviour of QCD see figure 1.10. Indeed we shall

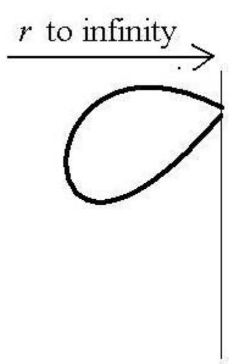

Figure 1.10: A local QCD operator extended in the AdS_5 (bulk)

calculate in this approach hadron spectra for mesons and baryons, form factors and even light cone wave functions. All this is done by solving classical equations – that is no loop calculations are performed– in the non-Euclidean geometry of AdS_5. Since there is here some special interest in light cone wave functions, I shall follow largely the lectures of Brodsky and Teramond [7] where these important quantities are derived from AdS/CFT. In the rest of the first part we have to learn dynamics in a non-Euclidean geometry, as a spin off we shall also learn a little bit of general relativity and cosmology.

1.2 Dynamics in non-Euclidean geometry

1.2.1 Basics of non-Euclidean geometry

We take here a very practical approach which leads us as fast as possible to the applications.

A scalar product contains all the information in Euclidean geometry: length and angle.

$$|\vec{a}| = \sqrt{a \cdot a}, \qquad \cos\theta = \frac{1}{|\vec{a}||\vec{b}|}\vec{a}\cdot\vec{b}$$

We write the scalar product in the clumsy form:

$$\vec{a}\cdot\vec{b} = \sum_{\mu\nu} e_{\mu\nu} a^\mu b^\nu$$

with

$$e_{\mu\nu} = \left\{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\}$$

in a three dimensional space.

The tensor $\{e_{\mu\nu}\}$ is called the **metric tensor** (or short metric) of Euclidean geometry. Since the length of a vector with components $\{dx^1, dx^2, \ldots dx^n\}$ is given by

$$(ds)^2 = \sum_{\mu\nu} e_{\mu\nu} dx^\mu\, dx^\nu \tag{1.2}$$

this expression is also referred to as "the metric".

In Minkowski space the metric tensor is given by :

$$\{\eta_{\mu\nu}\} = \left\{\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right\} \tag{1.3}$$

Since $\{\eta^{\mu\nu}\}$ is not positive definite, it is called a pseudo-Euclidean metric tensor and geometry in Minkowski space is called pseudo-Euclidean.

If $\{e_{\mu\nu}\}$ or $\{\eta_{\mu\nu}\}$ are not diagonal, but only symmetric, we can bring them by a change of the coordinate system into a diagonal form and by rescaling also to a form with ± 1 in the diagonals.

In non-Euclidean geometry, the elements of the metric are no longer constants, but may differ from point to point. The line element of eq. 1.2 therefore becomes:

$$(ds)^2 = \sum_{\mu\nu} g_{\mu\nu}(x)\, dx^\mu\ dx^\nu \tag{1.4}$$

where $g_{\mu\nu}$ is a symmetric matrix function, $g_{\mu\nu}(x) = g_{\nu\mu}(x)$.

If we want to determine the metric on a sphere in polar coordinates $\{\theta, \phi\}$ we do this best by considering a line element (see figure 1.11)

$$(ds)^2 = R^2 d\theta^2 + (R \sin\theta d\theta)^2$$

that is the metric tensor on a sphere S_2 is:

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}$$

R is a constant, but we see that the tensor depends indeed on the coordinate θ.

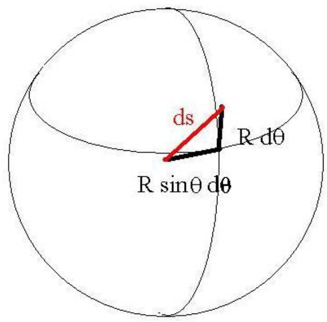

Figure 1.11: A line element on the sphere S_2

The metric of AdS

We can now make the statements made in the introduction mathematically precise.

The metric of the D3 brane in the 10 dimensional space of the Maldacena case is given by:

$$(ds_{10})^2 = \frac{1}{\sqrt{1 + L^4/r^4}} \left(-(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2\right) - \sqrt{1 + \frac{L^4}{r^4}}\,(dr^2 + r^2 d\Omega_5) \qquad (1.5)$$

here L is a constant which will be related to the parameters of the Yang Mills theory:

$$L^4 = g_{YM}^2 N_c (\alpha')^2$$

where g_{YM} is the gauge coupling constant, N_c is the number of colors and α' the inverse string tension (the slope of the trajectories in the Veneziano model (see (1.1); $d\Omega_5$ is the metric of a 5-dimensional sphere.

For large $r >> L$ this looks like a flat metric, but for small r we can write

$$(ds_{10})^2 \approx (ds_{\text{AdS}})^2 + d\Omega_5 \qquad (1.6)$$

with

$$(ds_{AdS})^2 = \frac{r^2}{L^2}(-(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2) - \frac{L^2}{r^2}(dr)^2 \qquad (1.7)$$

This is the metric of a 5-dimensional Anti-de-Sitter space. We see here directly the assertion made in the introduction: If we multiply the space-time coordinates x with α and divide the 5th coordinate r by α, the metric remains unchanged.

By coordinate transformations one can bring the metric in many forms. One which is very often used in pheneomenlogical applications is given by the replacement:

$$z = \frac{L^2}{r} \tag{1.8}$$

which yields

$$dz = -\frac{L^2}{r^2} dr$$

and therefore

$$(ds_{AdS})^2 = \frac{L^2}{z^2}\left(-(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 - (dz)^2\right) \tag{1.9}$$

Here we see very clearly the dilatation invariance: If all 5 coordinates are multiplied by a factor α the metric remains unchanged.

Another choice of coordinates is

$$r = e^{u/L} \tag{1.10}$$

in this case we obtain:

$$dr = \frac{1}{L} e^{u/L} du$$

and therefore

$$(ds_{AdS})^2 = \frac{e^{2u/L}}{L^2}(-(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2) - (du)^2 \tag{1.11}$$

So we see that things are very flexible in non-Euclidean geometry with general coordinate transformations. But there are still invariants and symmetries left, and the AdS space is indeed a maximally symmetric space. Before we understand that and can exploit it, we have to learn some more things.

Upper and lower indices

The inverse metric tensor, $\{g_{\mu\nu}\}^{-1}$ is denoted by $\{g^{\mu\nu}\}$, that is with **upper** indices:

$$\{g_{\mu\nu}\}^{-1} \cdot \{g^{\mu\nu}\} = \mathbf{1} \quad \text{thatis} \quad \sum_\nu g_{\mu\nu}\, g^{\nu\rho} \equiv g_\mu^\rho = \delta_\mu^\rho \tag{1.12}$$

where

$$\delta_\mu^\rho = \begin{matrix} 1 & \text{for} & \mu = \rho \\ 0 & \text{else} & \end{matrix} \tag{1.13}$$

From now on we shall use the Einstein convention: We sum over equal upper and lower indices, that is for instance:

$$g_{\mu\nu}\, g^{\nu\rho} \equiv \sum_\nu g_{\mu\nu}\, g^{\nu\rho} \tag{1.14}$$

where the summation runs over all coordinates (d in a d-dimensional space).

In Euclidean and pseudo-Euclidean (Minkowski-space) geometry one has:

$$\eta_{\mu\nu} = \eta^{\mu\nu} \tag{1.15}$$

as one easily verifies. **This is not the case for general non-Euclidean metric.**

A vector with upper indices is called a **contravariant** vector, one with lower indices is called a **covariant** vector.

A contravariant vector V^μ transforms under a coordinate transformation $x^\mu \to x'^\mu$ like:

$$V^\mu(x) \to V'^\mu = V^\nu \, \frac{\partial x'^\mu}{\partial x^\nu} \tag{1.16}$$

a covariant vector like

$$V_\mu(x) \to V'_\mu = V_\nu \, \frac{\partial x^\mu}{\partial x'^\nu} \tag{1.17}$$

We can transform contravariant vectors in covariant ones and vice versa (lower or rise indices) by the metric tensor:

$$g_{\mu\nu}(x)V^\nu = V_\mu; \qquad g^{\mu\nu}(x)V_\nu = V^\mu \tag{1.18}$$

The gradient of a scalar function $\phi(x)$, that is $\{\partial_\mu \phi\}$ is a covariant vector, as follows from the chain rule (we write ∂'_μ for $\frac{\partial}{\partial x'^\mu}$):

$$V'_\mu \equiv \partial'_\mu \phi'(x') = \partial'_\mu \phi(x) = \partial_\nu \phi(x) \frac{\partial x^\nu}{\partial x'^\mu} = V_\nu \, \frac{\partial x^\nu}{\partial x'^\mu}$$

which is the transformation law (1.17).

A covariant or contravariant or mixed tensor has several indices which transform according to (1.17 or 1.16) A covariant tensor of rank two e.g. transforms as:

$$V_{\mu\nu} \to V'_{\mu\nu} = V_{\rho\sigma} \, \partial'_\mu x^\rho \partial'_\nu x^\sigma \tag{1.19}$$

We can easily check, that the **derivative of a vector does not form a rank two tensor** (in contrast to special relativity).

$$\begin{aligned}
\partial'_\mu V'_\nu(x') &= \partial'_\mu \Big(V_\rho(x) \partial'_\nu x^\rho \Big) \\
&= \Big(\partial'_\mu V_\rho(x') \Big) \partial'_\nu x^\rho + V_\rho(x) \partial'_\mu \partial'_\nu x^\rho \\
&= \partial_\sigma V_\rho(x) \partial'_\mu x^\sigma \, \partial'_\nu x^\rho + V_\rho(x) \partial'_\mu \partial'_\nu x^\rho
\end{aligned} \tag{1.20}$$

We see that in order to get a rank two tensor we have to form

$$\partial'_\mu V'_\nu(x') - V_\rho(x) \partial'_\mu \partial'_\nu x^\rho . \tag{1.21}$$

This is a consequence of the fact, that in non-Euclidean geometry a vector changes its components under parallel transport. If we go from x_μ to $x_\mu + \epsilon$ we have

$$V_\nu(x) \to V_\nu(x + \epsilon) - \epsilon \, \Gamma^\rho_{\nu\mu} V_\rho(x)$$

where $\Gamma^\rho_{\mu\sigma}$ is called **parallel transporter**.

We define the covariant derivative:

$$D_\mu V_\nu \equiv \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho \tag{1.22}$$

$D_\mu V_\nu$ is a covariant rank 2 tensor.

One can show (using 1.20) that

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_k g_{\mu\nu}) \tag{1.23}$$

from which we see directly the symmetry:

$$\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} \tag{1.24}$$

Exercise

As a nice exercise, the result of which we need later, we calculate the two-dimensional Laplacian in polar coordinates. This is in Euclidean geometry, but since we do not use Cartesian coordinates, we can profitably use the machinery developed here.

In polar coordinates of the plane

$$x^1 = r\cos\phi \qquad x^2 = r\sin\phi \tag{1.25}$$

the line element is

$$(ds)^2 = (dr)^2 + (rd\phi)^2 \tag{1.26}$$

that is the metric tensor is

$$\{g\} = \begin{pmatrix} g_{rr} & g_{r\phi} \\ g_{r\phi} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad \{g\}^{-1} = \begin{pmatrix} g^{rr} & g^{r\phi} \\ g^{r\phi} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix} \tag{1.27}$$

Looking at (1.23) we see that only one contribution to the parallel transporters is different from zero, namely

$$\partial_r g_{\phi\phi} = 2r$$

Therefore we obtain easily from (1.23)

$$\begin{aligned} \Gamma^r_{rr} &= 0; \quad \Gamma^r_{r\phi} = 0; \qquad\qquad \Gamma^r_{\phi\phi} = -\frac{1}{2} g^{rr} \partial_r g_{\phi\phi} = -r \\ \Gamma^\phi_{rr} &= 0; \quad \Gamma^\phi_{r\phi} = \frac{1}{2} g^{\phi\phi} \partial_r g_{\phi\phi} = \frac{1}{r}; \quad \Gamma^\phi_{\phi\phi} = 0 \end{aligned} \tag{1.28}$$

The gradient of a scalar function $f(r,\phi)$ is

$$(\partial_r f, \partial_\phi f)$$

and therefore the covariant Laplacian:

$$\begin{aligned} D^2 f &= g^{rr} D_r \partial_r f + g^{\phi\phi} D_\phi \partial_\phi f \\ &= g^{rr} \Big(\partial_r^2 f - \Gamma^r_{rr} \partial_r f - \Gamma^\phi_{rr} \partial_\phi f \Big) + g^{\phi\phi} \Big(\partial_\phi^2 - \Gamma^r_{\phi\phi} \partial_r f - \Gamma^\phi_{\phi\phi} \partial_\phi f \Big) \\ &= \partial_r^2 f + \frac{1}{r} \partial_r f + \frac{1}{r^2} \partial_\phi^2 f \end{aligned} \tag{1.29}$$

This is the expression for the two dimensional Laplacian in polar coordinates (in Cartesian coordinates it is of course $(\partial_1^2 f + \partial_2^2 f)$.

1.2.2 Wave equations in non-Euclidean geometry

Scalar field

We start with the action of a free scalar field in Minkoswski space. It is given by:

$$A = \int d^4x \underbrace{\frac{1}{2}(\eta^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)}_{\mathcal{L}} \tag{1.30}$$

Variation of the action $\delta A = 0$ leads to the Euler-Lagrange equations:

$$\partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho\phi)} - \frac{\partial \mathcal{L}}{\partial\phi} = 0 \tag{1.31}$$

which leads to the wave equation (Klein-Gordon equation):

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + m^2\phi = 0. \tag{1.32}$$

If we go to non-Euclidean metric we have to replace:

- $\eta^{\mu\nu} \rightarrow g^{\mu\nu}$
- $d^4x \rightarrow d^4x\sqrt{|g|}$ (Square root of the determinant of the metric) This is an invariant volume element.
- $\partial \rightarrow D$ (covariant derivative) Fortunately for a scalar the covariant derivative is the normal one, see sect. 1.2.1.

Instead of the action (1.30) we obtain in non-Euclidean geometry:

$$A = \int d^4x \underbrace{\sqrt{|g|}\frac{1}{2}(g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)}_{\mathcal{L}} \tag{1.33}$$

and from (1.31 we obtain instead of 1.32 the wave function in non-Euclidean metric:

$$\partial_\rho\Big(\sqrt{|g|}g^{\rho\nu}\partial_\nu\phi\Big) + \sqrt{|g|}m^2\phi = 0. \tag{1.34}$$

or

$$g^{\rho\nu}(\partial_\rho\partial_\nu\phi) + m^2\phi = \frac{-1}{\sqrt{|g|}}\,\partial_\rho(\sqrt{|g|}g^{\rho\nu})\partial_\nu\phi \tag{1.35}$$

That is the interaction with the non-Euclidean metric (e.g. gravity) leads to an interaction term (r.h.s. of 1.35).

Electrodynamics in non-Euclidean geometry

As a small exercise, which will be useful later, we derive the Maxwell equations in non-Euclidean metric.

In electrodynamics we have

$$A = \int d^4x \underbrace{F_{\mu\nu}\, F_{\mu'\nu'} \eta^{\mu\mu'} \eta^{\nu\nu'}}_{\mathcal{L}_{em}} \tag{1.36}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.37}$$

and A_μ the electromagnetic vector potential.

The Euler-Lagrange equations are here:

$$\partial_\rho \frac{\partial \mathcal{L}_{em}}{\partial(\partial_\rho A_\sigma)} - \frac{\partial \mathcal{L}}{\partial A_\sigma} = 0 \tag{1.38}$$

From the variational principle $\delta A = 0$ follow the free Maxwell equations

$$\partial_\rho\, F^{\rho\sigma} = 0 \tag{1.39}$$

In a non-Euclidean metric the same replacements have to be done as in the scalar case. Here again we have not to care about the covariant derivatives because . Indeed if we insert covariant derivatives in (1.37) we obtain

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\mu A_\nu - (\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}) A_\sigma = \partial_\mu A_\nu - \partial_\mu A_\nu \tag{1.40}$$

because of the symmetry of the parallel transporters, see 1.24[2] The action for free electrodynamics in non-Euclidean geometry is therefore:

$$A = \int d^4x \underbrace{\sqrt{|g|} F_{\mu\nu}\, F_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'}}_{\mathcal{L}_{emg}} \tag{1.41}$$

The variation principle yields:

$$\partial_\rho \frac{\partial \mathcal{L}_{emg}}{\partial(\partial_\rho A_\sigma)} = \partial_\rho \frac{\partial}{\partial(\partial_\rho A_\sigma)} \Big(\sqrt{|g|} F_{\mu\nu}\, F_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'} \Big) = \partial_\rho \Big(\sqrt{|g|} g^{\mu\mu'} g^{\nu\nu'} \frac{\partial}{\partial(\partial_\rho A_\sigma)} (F_{\mu\nu}\, F_{\mu'\nu'}) \Big) \tag{1.42}$$

I give the calculation step by step, this is always advisable in non-Euclidean metrics, since there are many traps there.

$$\begin{aligned}
\frac{\partial \mathcal{L}_{emg}}{\partial(\partial_\rho A_\sigma)} &= \sqrt{|g|}\, g^{\mu\mu'} g^{\nu\nu'} \frac{\partial}{\partial(\partial_\rho A_\sigma)} \Big(\partial_\mu A_\nu - \partial_\nu A_\mu)\, 2\, \partial_{\mu'} A_{\nu'} \Big) \\
&= \sqrt{|g|}\, g^{\mu\mu'} g^{\nu\nu'} \Big(2\, \delta^\rho_\mu \delta^\sigma_\nu\, \partial_{\mu'} A_{\nu'} - 2\, \delta^\rho_\nu \delta^\sigma_\mu\, \partial_{\mu'} A_{\nu'} + 2 (\partial_\mu A_\nu - \partial_\nu A_\mu) \delta^\rho_{\mu'} \delta^\sigma_{\nu'} \Big) \\
&= 2\sqrt{|g|}\, g^{\rho\mu'} g^{\sigma\nu'} \partial_{\mu'} A_{\nu'} - 2\sqrt{|g|}\, g^{\sigma\mu'} g^{\rho\nu'} \partial_{\mu'} A_{\nu'} + \sqrt{|g|}\, g^{\mu'\rho} g^{\nu'\sigma} (\partial_{\mu'} A_{\nu'} - \partial_{\nu'} A_{\mu'}) \\
&= 4\sqrt{|g|}\, g^{\mu\rho} g^{\nu\sigma} (\partial_\mu A_\nu - \partial_\nu A_\mu)
\end{aligned} \tag{1.43}$$

[2] This result is general in Riemannian geometry. One calls such a antisymmetric construction a *form* and writes eq. 1.40 as $F = d_\wedge A$.

Inserting (1.43) into the result of the variational principle

$$\partial_\rho \frac{\partial \mathcal{L}_{emg}}{\partial(\partial_\rho A_\sigma)} = 0$$

we obtain as Maxwell equations in non-Euclidean geometry:

$$g^{\mu\rho} g^{\nu\sigma} \partial_\rho F_{\mu\nu} = \frac{-1}{\sqrt{|g|}} F_{\mu\nu} \partial_\rho (\sqrt{|g|}\, g^{\mu\rho} g^{\nu\sigma}) \tag{1.44}$$

Again we see that the non-Euclidean geometry leads to an interaction term. The left hand side acts like a current $J^\sigma = -F_{\mu\nu}\partial_\rho(\sqrt{|g|}\, g^{\mu\rho} g^{\nu\sigma})$

String action

In order to show the formal relation between non-Euclidean geometry and string theory, we also show the the classical action for a relativistic string. It is given by:

$$W = \int d\sigma\, d\tau \, \sqrt{||G||} \tag{1.45}$$

where $||G||$ is the determinant of the matrix $(G_{\alpha\beta})$ (the so called induced metric)

$$G = \begin{pmatrix} \partial_\sigma x^\mu \partial_\sigma x^\nu \eta_{\mu\nu} & \partial_\sigma x^\mu \partial_\tau x^\nu \eta_{\mu\nu} \\ \partial_\sigma x^\mu \partial_\tau x^\nu \eta_{\mu\nu} & \partial_\tau x^\mu \partial_\tau x^\nu \eta_{\mu\nu} \end{pmatrix}$$

This tensor G is just the metric tensor on the (2-dimensional) string.

1.3 The anti de Sitter space (AdS)

1.3.1 Maximal symmetry and conformal invariance

We have already shown the metric of the Anti-de-Sitter space (see 1.5 ff). Here we want to collect some more important properties.

The anti-de-Sitter space is a space of maximal symmetry, which means that as a d dimensional space it has $d\,(d+1)/2$ symmetries. Quite generally one can show, that a space with d dimensions has $d\,(d-1)/2$ isometries, that is symmetries which leave the metric invariant. If we look for the symmetries of 4-dimensional Minkowski space, we have besides the $4 \cdot 3/2 = 6$ Lorentz transformations another 4 symmetry transformations, namely the translations in space and time, that is we have indeed $5 \cdot 4/2 = 10$ symmetry transformations.

A non-Euclidean maximally symmetric space can be constructed from a $d+1$ dimensional space by eliminating one dimension by an invariant condition. Than this resulting d-dimensional space has still the $(d+1)\,d/2$ symmetries of the $(d+1)$-dimensional space. An intuitive example is the sphere S_2. If we start with a 3-dimensional space it has 3 isometries, the 3 rotations. A sphere is constructed in a 3-dimensional space by the condition $x^2 + y^2 + z^2 = R^2$ which is invariant under rotations. Hence the surface of the sphere ($d = 2$) has still the $3 \cdot 2/2 = 3$ symmetries of the 3-dimensional space. The Euclidean 2-dimensional plane is a limit of the sphere with the radius going to infinity.

We shall consider a 5-dimensional anti-de-Sitter space, hence there are 15 symmetry transformations. On the other hand it is said to be dual to a Yang Mills theory in the four dimensional Minkowski space, which as a four-dimensional maximally symmetric space has only 10 symmetries. Therefore the special Yang Mills theory dual to AdS_5 must have additional symmetries, and these are in the Maldacena case the so called conformal symmetries. This explains the name fully: AdS/CFT : AdS for anti-de-Sitter space, CFT for conformal field theory.

A conformal symmetry is the symmetry under:
•Dilatation : $x \to \lambda x$
• Inversion : $x^\mu \to \frac{a\, x^\mu}{x^2}$
We deal here only with dilatations, since they lead in principle to the inversions.

We expect that a theory where the Lagrangian contains no scale (quantity with dimension) the theory is scale invariant, that is invariant under dilatations.

In a somewhat sketchy way one can see this easily for gauge theories:

We replace: $x \to x' = \lambda x$ then $\frac{\partial}{\partial x^\mu} \to \frac{1}{\lambda}\frac{\partial}{\partial x^\mu}$ and $A_\mu(x) \to A'_\mu(x') = A'_\mu(\lambda x)$

If we set $A'_\mu(\lambda x) = \frac{1}{\lambda} A_\mu(x)$, then $F'_{\mu\nu}(\lambda x) = \frac{1}{\lambda^2} F_{\mu\nu}(x)$

For Abelian gauge theories this is trivial, but it holds also for non-Abelian gauge theories where we have:

$$F^C_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i \sum_{AB} f^{ABC} A^A_\mu A^B_\nu \tag{1.46}$$

Each derivative ∂_μ and each field A_μ leads to a factor $\frac{1}{\lambda}$. Therefore the action remains invariant under dilatation:

$$A = \int d^4x\, F_{\mu\nu} F_{\mu'\nu'} \eta^{\mu\mu'} \eta^{\nu\nu'} \to A' = \int d^4(\lambda x) \frac{1}{\lambda^4} F_{\mu\nu} F_{\mu'\nu'} \eta^{\mu\mu'} \eta^{\nu\nu'} = A \tag{1.47}$$

The same holds for a massless ϕ^4 theory, with $\phi'(\lambda x) = \frac{1}{\lambda}\phi(x)$ But if there is a mass term, dilatation invariance is no longer valid.

$$A = \int d^4x\, \eta^{\mu\mu'} \partial_\mu \phi \partial_{\mu'} \phi - m^2 \phi^2 - g\phi^4 \to A' = \int d^4x\, \eta^{\mu\mu'} \partial_\mu \phi \partial_{\mu'} \phi - \lambda^2 m^2 \phi^2 - g\phi^4 \neq A \tag{1.48}$$

The supersymmetric Yang Mills theory in the Maldacena case is indeed conformal invariant and there is a one-to-one correponcence between the symmetries of AdS_5 and CFT. This is one of the reasons for the conjecture.

QCD, however is not supersymmetric and is not conformal invariant as a **quantum theory**. In the renormalization procedure one has to include a mass scale which breaks this invariance and leads among other things to the running coupling with a dimensional quantity, Λ_{QCD}:

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \, \log(Q^2/\Lambda^2_{\mathrm{QCD}}}) + \ldots \qquad \beta_0 = 11 - \frac{2}{3} n_f$$

One has argued, that from some phenomenological analysis follows that for large values of α_s (large distances) the coupling remains fixed and therefore there might be approximate conformal invariance. But on the other hand, for confinement we need a dimensional quantity (the rise of the linear potential, e.g.) and therefore the maximal symmetry of AdS_5 must be violated somehow. We shall see this explicitly in the phenomenological part.

The relation between dilatation symmetry and the symmetries in AdS_5 can be seen directly from the metric:

$$(ds)^2 = \frac{L^2}{z^2}(-(dx_1)^2 - (dx_2)^2 - (dx_3)^2 + (dx_4)^2 - (dz)^2). \tag{1.49}$$

This metric is invariant under a dilatation of all coordinates $\{x, z\} \to \{\lambda x, \lambda z\}$.

From this we see further, that the variable z is something like a scaling variable in Minkowski space. A small value for λ corresponds to small scales, that is small distances and correspondingly high energies. If by choosing a small scale λ we investigate the small x behavior, also z is small, because it is multiplied by the same factor λ.

1.3.2 Parallel transporters and curvature in AdS

In this subsection we collect some results for the special AdS_5 geometry.

We first fix the notation for further use. We introduce the 5-dimensional pseudo-Euclidean (Minkowski) tensor:

$$\{\eta_{AB}\} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.50}$$

The indices A, B, C run from 1 to 5. In accordance with most of the literature we call the 5th variable $x^5 \equiv z$ and use $\partial_5 \equiv \partial_z$.

The metric tensor is then, see (1.9)

$$\{g_{AB}\} = \frac{L^2}{z^2}\{\eta_{AB}\} \qquad \{g_{AB}\}^{-1} = \{g^{AB}\} = \frac{z^2}{L^2}\{\eta^{AB}\} \tag{1.51}$$

From 1.51 follows $\partial_K g_{LN} = -2\frac{L^2}{z^3}\delta^5_K \eta_{LN}$ and the parallel transporters, see 1.23, can be easily calculated to

$$\begin{aligned} \Gamma^L_{MN} &= \frac{1}{2} g^{LK}\left(\partial_M g_{KN} + \partial_N g_{KM} - \partial_K g_{MN}\right) \\ &= \frac{-1}{z}\left(\delta^5_M \delta^L_N + \delta^5_N \delta^L_M - \eta^{L5}\eta_{MN}\right) \end{aligned} \tag{1.52}$$

We shall use these parallel transporters later for the covariant derivative which is necessary for the treatment of particles with higher spin.

As we have seen from equations 1.7-1.51 general coordinate transformations allow many different forms of the metric for the same geometry. Are there also intrinsic invariants. The answer is yes, the scalar curvature e.g..

We first give some results which are generally valid: The **curvature tensor** $R^\lambda_{\mu\nu\kappa}$ can be calculated from the metric tensor. It is most convenient to express it in terms of the parallel transporters :

$$R^\lambda_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} \tag{1.53}$$

The **Ricci tensor** is obtained by contracting over the contravariant with the second covariant index:

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \tag{1.54}$$

The **scalar curvature** as

$$R = g^{\mu\nu} R_{\mu\nu} \tag{1.55}$$

Now we come to the special case AdS_5. The curvature R can be calculated from 1.55 together with 1.53 and 1.52 to:

$$R = \eta^{55}(5^2 - 5)\frac{1}{L^2} = -\frac{20}{L^2} \tag{1.56}$$

For a maximally symmetric space the curvature tensor is particularly simple :

$$R_{MNRS} = \frac{1}{L^2}(g_{NR}\, g_{MS} - g_{NS}g_{MR}) \tag{1.57}$$

which means that the curvature tensor is determined by one single constant, the constant curvature. You can check this by a simple but tedious calculation for AdS_5.

1.3.3 Short excursion to general relativity and cosmology

This section is not needed for the following. It is only intended for those not familiar with the elements of general relativity.

Here we stay in four dimensions. The action for pure gravity in Einsteins theory is very simple:

$$A = \int d^4x \sqrt{|g|} R$$

where $|g|$ is the modulus of the determinant of the metric, it is necessary, since only $d^4x\sqrt{|g|}$ is invariant under general coordinate transformations.

Coupling to matter leads to the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, R = -8\pi G_N T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.58}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter, G_N Newton's gravitational constant and Λ the famous cosmological constant. In the original work Einstein did not include a cosmological constant (see the Einstein bust in front of the Department), but he later introduced it in order to allow a steady state universe. When the expansion of the universe became apparent through he experiments of Hubble, Einstein considered the introduction of the cosmological constant as his biggest blunder. Modern observational data indicate however that there is indeed a cosmological constant. It is necessary to explain the fast expansion of the universe.

The energy momentum tensor can be obtained from the total Lagrangian, including matter, by variation with respect to the metric:

$$T_{\mu\nu} = \frac{\delta \mathcal{L}_{\rm tot}}{\delta g^{\mu\nu}}. \tag{1.59}$$

For radiation, which was dominant in the early stage of the universe, it is the Maxwell energy momentum tensor, for "cold classical matter" it is

$$T_{\mu\nu} = \rho\, u_\mu u_\nu \tag{1.60}$$

where u_μ is the 4-velocity.

The de-Sitter metric in 4 dimensions is

$$(ds)^2 = \frac{L^2}{z^2}\big(- (dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dz)^2\big) \tag{1.61}$$

Note that in the de-Sitter metric the"time" direction is singled out, in contrast to the anti-de-Sitter metric, where a "space" component is singled out. The Ricci tensor is

$$R_{\mu\nu} = \frac{-3}{L^2} g_{\mu\nu} \tag{1.62}$$

and the curvature scalar is

$$R = \frac{-12}{L^2}. \tag{1.63}$$

Now we see that the de-Sitter space in four dimensions is a solution of the Einstein equations 1.58 with $T_{\mu\nu} = 0$ (no matter) and a cosmological constant $\Lambda = \frac{3}{L^2}$.

If we introduce new coordinates $z = e^{-ct/L}$ then the metric has the familiar form with time singled out:

$$(ds)^2 = (c\,dt)^2 - L^2 e^{2ct/L}((dx^1)^2 + (dx^2)^2 + (dx^3)^2) \tag{1.64}$$

The factor $L^2 e^{2ct/L}$ in front of the space part of the metric can be interpreted as the unit length. If $L > 0$ we have an expanding universe **without matter**.

Though matter is rather diluted in the universe, we cannot ignore it completely. Fortunately in cosmology one can make use of the cosmological principle: Averaged over very large (cosmological) scales, matter is evenly distributed and no point is preferred, i.e. $\rho =$ constant, $T_{00} = \rho$, and all other components are zero. This leads to a metric with a maximally symmetric subspace of three dimensions, the Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - \mathcal{R}(t)^2 \left(\frac{(dr)^2}{1 - kr^2} + r^2 d\Omega \right). \tag{1.65}$$

If $k = 0$, the universe is asymptotically flat, for $k \neq 0$ it is either finite ($k > 0$) or infinite ($k < 0$). From microwave data one has rather good indications that $k = 0$. Which we shall assume in the following.

If the Robertson Walker metric is inserted into the Einstein equations one obtains after a straightforward calculation:

$$\left(\frac{\partial_t \mathcal{R}(t)}{\mathcal{R}(t)} \right)^2 = \frac{1}{3}(8\pi G_N \rho + \Lambda) \tag{1.66}$$

and

$$\frac{\partial_t^2 \mathcal{R}(t)}{\mathcal{R}(t)} = \frac{1}{3}\Big(\Lambda - 4\pi G_N(\rho + 3p) \Big) \tag{1.67}$$

where ρ is the density and p the pressure of matter. The $\mathcal{R}(t)$ has nothing to do with the curvature, but is rather the unit length. If $\Lambda \neq 0$ the metric of the universe will approach more and more a de-Sitter metric, since in the expanding universe the matter density ρ decreases.

According to modern observational data, $\Lambda > 0$ and $\mathcal{R}(t)$ is increasing. If we extrapolate back using classical theory of (Einstein) gravitation, $\mathcal{R}$ was zero some $14\,10^9$ years ago. This is the famous big bang, since for that time $\mathcal{R}(t_0) = 0$ the universe had no extension. It should be noted however that this extrapolation to $\mathcal{R}(t_0) = 0$ is not justified, since for very small values of $\mathcal{R}$ quantum effects of gravitation are supposed to be dominant. Superstring theory is just the attempt, to calculate these effects. In this theory a minimal length scale exists and therefore it is very well possible that there was no big bang but only a big bounce.

Chapter 2

Applications

We now come to the applications of AdS/CFT. We shall first try to calculate particle spectra. The general procedure is the following: We look for such a solutions of the wave equation for a field in AdS_5 which is the product of a free field in Minkowski space multiplied by a function depending on the fifth variable z, that is

$$\Phi(x,z) = \epsilon(p) e^{-ipx}\,\phi(z), \tag{2.1}$$

where p is the 4-momentum and $\epsilon(p)$ a polarization vector or spinor of the free field. To say it in a very high-brow way: The full particle field in 4-dimensional Minkowski space M_4 is the holographic picture of the field $\Phi(x,z)$ ind the 5-dimensional AdS. If AdS/CFT yielded confinement, we obtained solutions only for discrete values of $p^2 = m^2$, corresponding to the hadronic states. But it will turn out, that this is not the case. Therefore the AdS_5 metric has to be modified in phenomenological applications.

In this second part I shall proceed very slowly, dwelling not only on the principal questions but also on the technical ones.

A remark on **notation**: We shall use capital Latin indices A,B C,... K,M,L ... in AdS_5 and Greek indices $\alpha, \beta, \ldots \kappa, \lambda, \mu \ldots$ in the 4-dimensional border space M_4. So we write e.g.:

$$\eta^{AB}\partial_A\partial_B\Phi(x,z) = (\eta^{\alpha\beta}\partial_\alpha\partial_\beta - \partial_z^2)\Phi(x,z) \tag{2.2}$$

It is customary to call the extension in the 5'th dimension of AdS_5 *bulk* and the 5th variable z the *holographic* variable, since the Yang-Mills theory in the 4-dimensional space can be viewed as a hologram of the theory in 5 dimensions.

2.1 The scalar particles

We shall proceed in 5 steps:
1) Derive the wave equation
2) Discuss general properties of the solutions
3) Solve the wave equation
4) Modify pure AdS_5
5) Compare wit data

2.1.1 Action and wave equation

We now use the results of section 1.2.2. We start with a Lagrangian for a scalar (or pseudoscalar) field $\Phi(x, z)$. Here x refers to the 4 Minkowski space-time coordinates, z is the 5th coordinate in AdS_5. Since scalar particles are no natural ingredients in a gauge theory we allow for a term without derivatives, $\mu^2\Phi^2$, which looks like a mass in the 5 dimensional space. The value of μ^2 will be fixed later, it should be noted that is is just a parameter and has nothing to do with masses in Minkowski space.

Our Lagrangian is thus

$$\mathcal{L} = \sqrt{|g|}\frac{1}{2}(g^{MN}\partial_M\Phi\partial_N\Phi - \mu^2\Phi^2) \tag{2.3}$$

In AdS_5 we have (see 1.51)

$$\{g_{AB}\} = \frac{L^2}{z^2}\{\eta_{AB}\} \qquad \{g_{AB}\}^{-1} = \{g^{AB}\} = \frac{z^2}{L^2}\{\eta^{AB}\} \tag{2.4}$$

It is convenient for further calculations to write

$$\frac{L^2}{z^2} \equiv e^{2A(z)} \quad \text{with } A(z) = -\log z + \log L \tag{2.5}$$

We then have

$$\sqrt{|g|} = \sqrt{\left\| \begin{matrix} -e^{2A(z)} & 0 & 0 & 0 & 0 \\ 0 & -e^{2A(z)} & 0 & 0 & 0 \\ 0 & 0 & -e^{2A(z)} & 0 & 0 \\ 0 & 0 & 0 & e^{2A(z)} & 0 \\ 0 & 0 & 0 & 0 & -e^{2A(z)} \end{matrix} \right\|} = e^{5A(z)} \tag{2.6}$$

and

$$g_{MN} = e^{2A(z)}\eta_{MN}; \quad g^{MN} = e^{-2A(z)}\eta^{MN} \quad \mathcal{L} = \frac{1}{2}e^{3A(z)}\left(\eta^{MN}\partial_M\Phi\,\partial_N\Phi - e^{2A(z)}\mu^2\Phi\right) \tag{2.7}$$

The ingredients of the Euler-Lagrange equations

$$\partial_A\frac{\partial\mathcal{L}}{\partial(\partial_A\Phi)} - \frac{\partial\mathcal{L}}{\partial\Phi} = 0 \tag{2.8}$$

are

$$\partial_A\frac{\partial\mathcal{L}}{\partial(\partial_A\Phi)} = \partial_A\Big(e^{\kappa A(z)}\eta^{AB}\partial_B\Phi\Big); \quad \frac{\partial\mathcal{L}}{\partial\Phi} = -\mu^2\Phi e^{(\kappa+2)A(z)}; \quad \kappa = 3 \tag{2.9}$$

Since we have to deal with similar expressions in the following, the 3 in the factor $e^{3A(z)}$ is replaced by κ. We obtain as wave equation:

$$e^{\kappa A(z)}\Big(\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Phi - \partial_z^2\Phi - \kappa\partial_z A(z)\partial_z\phi + \mu^2 e^{2A(z)}\Phi\Big) = 0. \tag{2.10}$$

(The greek letters run from 1 to 4, that is $\eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\partial_1^2 - \partial_2^2 - \partial_3^2 + \partial_4^2$ is the usual d'Alembert 0perator in 4 space-time dimensions (see 2.2)).

As mentioned above we make the ansatz (2.1 with $\epsilon(p) = 1$ for a scalar field):

$$\Phi(x, z) = e^{-ipx}\phi(z) \tag{2.11}$$

and obtain for $\phi(z)$ the equation :

$$-\partial_z^2\phi - \kappa(\partial_z A)\partial_z\phi + e^{2A(z)}\mu^2\phi = p^2\phi \tag{2.12}$$

In principle we hope to obtain the mass of the scalar particle in 4 dimensions from that equation, namely discrete values of $m^2 = p^2$.

In order to get some better insight in the structure of the solutions, we shall bring this equation into the form of a 1-dimensional Schroedinger equation:

$$-\partial_z^2\psi + V(z)\psi = m^2\psi \tag{2.13}$$

For that we have in 2.12 to get rid of the single derivative term, $-\kappa(\partial_z A)\partial_z\phi$ We introduce

$$\phi(z) = e^{g(z)}\psi(z), \tag{2.14}$$

where we can choose $g(z)$ in such a way as to cancel the linear derivative. From product differentiation we easily obtain:

$$\begin{aligned} \partial_z\phi &= e^g(\partial_z g\psi + \partial_z\psi) \\ \partial_z^2\phi &= e^g\Big((\partial_z^2 g + (\partial_z g)^2)\psi + 2\partial_z g\,\partial_z\psi + \partial_z^2\psi\Big) \end{aligned} \tag{2.15}$$

Inserting 2.15 into 2.12 we obtain, after dividing by e^g:

$$-\partial_z^2\psi - (2\partial_z g + \kappa\partial_z A)\partial_z\psi - \Big(\partial_z^2 g + (\partial_z g)^2 + \kappa(\partial_z A)\partial_z g\Big)\psi + e^{2A}\mu^2\psi = m^2\psi \tag{2.16}$$

If we insert

$$\partial_z g = -\frac{\kappa}{2}\partial_z A \tag{2.17}$$

we see that indeed the linear derivative vanishes and we obtain finally

$$-\partial_z^2\psi + \Big(\frac{\kappa}{2}\partial_z^2 A + (\frac{\kappa}{2}\partial_z A)^2\Big)\psi + e^{2A(z)}\mu^2\phi = m^2\phi \tag{2.18}$$

That is we have the form of eq. 2.13 with the potential

$$V(z) = \Big(\frac{\kappa}{2}\partial_z^2 A + (\frac{\kappa}{2}\partial_z A)^2\Big) + e^{2A(z)}\mu^2. \tag{2.19}$$

If we insert $A(z) = -\log z + \log L$ (see eq. 2.5) we obtain for the potential in 2.13

$$V(z) = \frac{1}{4z^2}(4\mu^2L^2 + (\kappa+1)^2 - 1) \tag{2.20}$$

and

$$\phi(z) = e^{g(z)}\psi(z) = (z/L)^{\kappa/2}\psi(z). \tag{2.21}$$

It should be emphasized, that the equation 2.13 which looks like a Schrödinger equation is the result of a fully relativistic procedure.

Unfortunately this potential 2.20 vanishes for $z \to \infty$, it is therefore not confining. Therefore we have to modify the metric in the AdS_5. There are two major strategies: One is to introduce a "hard wall", that is to confine the space to values $z \leq z_0$, where $z_0 \approx 1/\Lambda_{QCD}$ seems to be a natural choice. This is the **hard-wall model**. It is simple, but has certain deficiencies which however can be cured by another model. In that model an additional field is introduced which breaks the maximal symmetry in AdS_5, and hence the conformal symmetry in QCD. This is the so called **soft-wall model**.

2.1.2 Scaling behaviour

Before we come to explicit solutions, we shortly discuss the behaviour of the solution for small values of z. I had mentioned in section 1.3.1 that z is like a scaling variable. Small scales in QCD correspond to small values of z. For small scales (that is small distances, high energies) dilation symmetry is valid in QCD and therefore the Field Φ should have the same scaling behaviour in z as the corresponding QCD operator.

A fermion field in a 4-dimensional space has the conformal dimension 3/2 (see sect. 1.3.1), that is it scales like [1] $\psi \to \lambda^{-3/2}\psi$. A scalar field in QCD is composed of a quark and an antiquark field, therefore its conformal dimension is 3. We therefore expect that[2] $\Phi(x,z) \sim z^3$.. This is assumed in many older papers. But in the newer papers normally one assumes that the scaling is not given by the **canonical conformal dimension**, but by what is called the **"twist dimension"**. The twist is conventionally defined as canonical dimension minus spin, that is for the quark-antiquark operator product $\bar\psi(x)O\psi(x)$, with $O = 1$ or $O = \gamma_5$, the twist dimension is $3 - 2\,\frac{1}{2} = 2$. Therefore in newer papers a behaviour $\Phi \sim z^2$ for scalar operators in AdS$_5$ which correpond to scalar QCD operators is generally assumed. Another argument for this scaling behaviour will be given in the section 2.2.

In order to find the scaling behaviour of the solutions of 2.13 we make a power series expansion of $\psi(z)$ starting with z^ℓ:

$$\psi(z) = z^\ell + O(z^{\ell+1}) \tag{2.22}$$

and obtain then for the small z behaviour of ψ from eq. 2.13 :

$$(-\ell(\ell-1) + V(z))z^{\ell-2} + O(z^{\ell-1}) = 0 \tag{2.23}$$

and with the potential 2.20 we obtain:

$$\psi \sim z^\ell \quad \text{with} \quad -\ell(\ell-1) + \frac{(4\mu^2L^2 + (\kappa+1)^2 - 1}{4} = 0 \tag{2.24}$$

that is

$$\ell = \frac{1}{2} + \sqrt{\mu^2L^2 + \kappa + 1} \tag{2.25}$$

From eq. 2.14,2.17 and 2.21 we obtain for the anti-de-Sitter metric:

$$\phi \sim e^{g(z)}\psi(z) \sim z^{\kappa/2}\psi(z) \sim z^{(\kappa+1)/2+\sqrt{\mu^2L^2+\kappa+1}}. \tag{2.26}$$

For scaling with dimension ℓ_d we must therefore have

$$2 + \sqrt{\mu^2L^2 + \kappa + 1} = \ell_d \qquad \text{that is} \quad \mu^2L^2 = \ell_d(\ell_d - \kappa - 1) \tag{2.27}$$

For scaling according to canonical conformal dimension $\ell_d = \kappa$ and therefore $\mu^2L^2 = -\kappa$, for scaling according to "twist dimension" ($\ell_d = 2$) we have $\mu^2L^2 = -\kappa - 1$. From string theory one has a stability condition, namely that $\mu^2L^2 \geq -4$ (Breitenlohner-Maison bound). For scaling with twist dimension we just exhaust this stability limit, since in the case considered here we have $\kappa = 3$.

[1]Then the action for am massless fermion, $A = \int d^4x\,\bar\psi(x)\gamma^\mu\partial_\mu\psi(x)$ is dilatation invariant
[2]In that case $\Phi'(\lambda z) = \frac{1}{\lambda^3}(\lambda z)^3 = \Phi(z)$

2.1.3 Solution

Equations with $1/y^2$ potentials are likely to be brought into the form of Bessel's differential equation. The generic form of the Bessel equation is

$$\Big(-\partial_y^2 - \frac{1}{y}\partial_y + \frac{\nu^2}{y^2}\Big) J_\nu(y) = J_\nu(y); \tag{2.28}$$

the solution $J_\nu(y)$ is the Bessel function of order ν (BesselJ$[\nu, z]$ in mathematica). For small y it behaves as

$$J_\nu(y) = y^\nu \Big(1 + O(\nu^2)\Big). \tag{2.29}$$

(There is another solution which is singular at $y = 0$ which does not concern us here.) We bring by the same procedure as discussed before the Bessel equation into the form of a Schrödinger eq. 2.13 by introducing

$$J_\nu(y) = y^{-1/2}\,\psi_\nu(y) \tag{2.30}$$

Inserting that into the Bessel eq. 2.28 we obtain for $\psi(z)$ the equation:

$$(-\partial_y^2 + \frac{4\nu^2 - 1}{4y^2})\psi(y) = \psi(y) \tag{2.31}$$

This is close to our equation 2.13. Indeed, if we put $y = zm$ we obtain after multiplication with m^2:

$$(-\partial_z^2 + \frac{4\nu^2 - 1}{4z^2})\psi(mz) = m^2\psi(mz) \tag{2.32}$$

and we obtain our equation 2.13 with the potential 2.20 if we put

$$\nu^2 = \mu^2 L^2 + (\kappa + 1) \tag{2.33}$$

The final solution for the scalar field is thus given by, see 2.30 and 2.21:

$$\psi(z) \sim z^{1/2} J_\nu(mz); \quad \phi(z) \sim e^{g(z)}\psi(mz) \sim z^{(\kappa+1)/2} J_\nu(mz) \tag{2.34}$$

As expected, this equation allows a solution for any m and therefore does not yield a discrete hadron spectrum. Therefore we have to modify somehow the pure AdS$_5$ theory.

2.1.4 The hard wall model

In this model the AdS$_5$ is confined to values

$$0 \le z \le z_0 \tag{2.35}$$

and one demands that at the "hard wall" the function $\phi(z)$ vanishes[8, 9, 10]:

$$\phi(z_0) = 0. \tag{2.36}$$

We impose this boundary condition on the solutions of 2.12, that is:

$$\phi(z_0) = z_0^{(\kappa+1)/2} J_\nu(z_0) = 0 \tag{2.37}$$

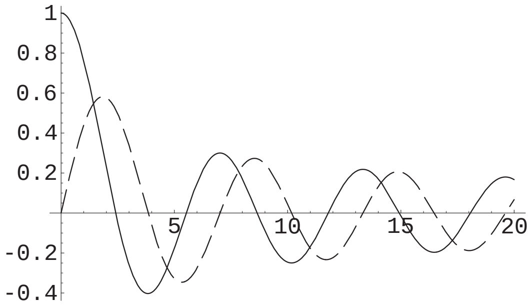

Figure 2.1: Bessel functions $J_1(x)$ (solid) and $J_2(x)$ (dashed)

ν	$j_{\nu,1}$	$j_{\nu,2}$	$j_{\nu,3}$	$j_{\nu,4}$	$j_{\nu,1}$	$j_{\nu,2}$	$j_{\nu,3}$	$j_{\nu,4}$
	approx.				exact			
0	2.41	5.52	8.65	11.79	2.40	5.52	8.65	11.79
1	3.83	7.02	10.17	13.32	3.83	7.02	10.17	13.32
2	5.16	8.42	11.62	14.80	5.14	8.42	11.62	14.80
3	6.45	9.78	13.02	16.22	6.38	9.76	13.02	16.22
4	7.73	11.11	14.39	17.63	7.59	11.06	14.37	17.62
5	9.00	12.42	15.74	19.00	8.77	12.34	15.70	18.98

Table 2.1: The 1.,2.,3. and 4. zero of Bessel functions J_ν as calculated with equation 2.39 and the exact values

This boundary condition 2.36 selects discrete values of m determined by the zeros of the Bessel functions (see figure 2.1). Denoting by $j_{\nu,s}$ the s'th zero of the Besselfunction $J_\nu(x)$ we obtain the relation

$$m = j_{\nu s}/z_0 \tag{2.38}$$

The value of ν

The zeros of the Bessel functions can for not so high ν quite well be approximated by

$$j_{\nu,s} \approx (s + \frac{\nu}{2} - \frac{1}{4})\pi - \frac{4\nu^2 - 1}{8\pi(s + \frac{\nu}{2} - \frac{1}{4})} - \ldots. \tag{2.39}$$

The first 4 zeros calculated with this formula, together with the exact numbers are given in table 2.1 for $\nu = 1 \ldots 5$.

We see that a general feature of the string picture is realized: We obtain not only one scalar (pseudoscalar) meson, but a whole series, corresponding to the zeros of the Bessel function with the same index ν. These higher resonances correspond to string excitations, as they are also seen in the Veneziano model as daughter trajectories (see figure 1.2).

Comparison with experiment

Although the pion plays a very special role in QCD as the pseudo-Goldstone boson of spontaneously broken chiral symmetry, we start with the discussion of pseudoscalar particles[3] . Besides the pion at 140 Mev/c^2 there are radial excitations with the same quantum number as the pion at 1320 and 1812 MeV /c^2. It is certainly hopeless to reproduce the small mass of the pion, since the pion is as Goldstone Boson a special case, but with $z_0^{-1} \approx \Lambda_{\rm QCD} \approx 300$ MeV (for 3 flavors)

[3]In order to treat this problem more realistically, in [9] and [10] a much richer field containt in AdS$_5$ is assumed

Experiment	π 140MeV	π 1300 ± 100MeV	π 1812 ± 14MeV
$\nu = 1$; $z_0^{-1} = 167$ MeV	639	1173	1700
$\nu = 0$; $z_0^{-1} = 207$ MeV	501	1140	1797

Table 2.2: Experimental and hard-wall masses of the pion resonances. $\nu = 1$ corresponds to canonical scaling, $\nu = 0$ to scaling with the "twist dimension" ($\ell_d = 2$).

Experiment				
f_0	980 $\pm$10 MeV	1400 $\pm$100 MeV	1505 $\pm$6MeV	1724 $\pm$7 MeV
a_0	980 $\pm$1.2 MeV	1474 $\pm$19 MeV		
$\nu = 1; z_0^{-1} = 150$MeV	576	1055	1529	2003
$\nu = 0; z_0^{-1} = 174$MeV	418	958	1502	2074

Table 2.3: Experimental values and a fit of hard-wall masses to the f_0 /a_0 masses

also the masses of the radial excitations are far to high. On the other hand there is no a priori reason to set z_0^{-1} exactly equal to $\Lambda_{\rm QCD}$ and so we may fit z_0^{-1}.

In table 2.2 we display fits to the masses for the choice $\nu = 1$ (canonical scaling) and $\nu = 0$ (twist scaling) according to a least square fit. The agreement is satisfactory.

It is tempting to assume that the resonances corresponding to the different values of ν correspond to orbital excitations. Then we would get a similar pattern as the Veneziano model. There are however crucial differences: 1) The radial and the supposed orbital exciatations are not linear in m^2, but approximately linear in m. 2) The "daughters" have not the same mass as the mothers since eg. $j_{0,2} \neq j_{1,1}$. These deficiencies are not present in the soft wall model.

The weakness of the linear increase is especially to be felt in the case of the scalar mesons f_0 (I=0, J=0, P=+) and a_0 (I=1, J=0, P=+), where 4 resonances have been observed. This can be seen from table 2.3.

2.1.5 The soft wall model

This model was proposed by Katz et. al[11]. Here an additional field is inserted, which breaks the dilatation symmetry (called dilaton field). That is, instead of the Lagrangian 2.3 we have:

$$\mathcal{L} = e^{-D(z)}\sqrt{|g|}\frac{1}{2}(g^{MN}\partial_M\Phi\partial_N\Phi - \mu^2\Phi^2) = \frac{1}{2}e^{\kappa A(z)-D(z)}(\eta^{MN}\partial_M\Phi\partial_N\Phi - \mu^2 e^{2A(z)}\Phi^2) \tag{2.40}$$

We see that $e^{\kappa A(z)}$ has been replaced by $e^{\kappa A(z)-D(z)}$. Following the same procedure as in section 1.2.2 we come to the wave equation:

$$e^{\kappa A(z)-D(z)}\Big(\eta^{\alpha\beta}\partial_\alpha\partial_\beta\Phi - \partial_z^2\Phi - (\kappa\partial_z A(z) - \partial_z D(z))\partial_z\phi + \mu^2 e^{2A(z)}\Phi\Big) = 0 \tag{2.41}$$

instead of 2.10 and to

$$-\partial_z^2\phi - (\kappa\partial_z A - \partial_z D)\partial_z\phi + e^{2A(z)}\mu^2\phi = p^2\phi \tag{2.42}$$

instead of 2.12. That is we indeed had to only to replace in equations 2.15 to 2.19

$$\kappa\partial_z A(z) \to \kappa\partial_z A(z) - \partial_z D(z) \tag{2.43}$$

.

We then obtain the Schroedinger-like wave equation 2.13 with the potential:

$$V(z) = \Big(\frac{1}{2}(\kappa\partial_z^2 A - \partial_z^2 D) + (\frac{1}{4}(\kappa\partial_z A - \partial_z D)^2\Big) + e^{2A(z)}\mu^2. \tag{2.44}$$

If we insert $A(z) = -\log z + \log L$ (see eq. 2.5) and set $D(z) = \lambda^2 z^2$ we obtain for the potential in 2.13:

$$V(z) = \frac{1}{4z^2}(4L^2\mu^2 + (\kappa+1)^2 - 1) + \lambda^4 z^2 + (\kappa - 1)\lambda^2 \tag{2.45}$$

and

$$\phi(z) = e^{g(z)}\psi(z) = (z/L)^{\kappa/2}\, e^{\lambda^2 z^2/2}\psi(z). \tag{2.46}$$

Here we see that the potential 2.45 is rising with z^2, it looks like an oscillator potential and we have good reasons to expect a discrete spectrum for m^2.

Before we compare with experiment, we shall shortly discuss the harmonic oscillator in 2 dimensions.

Harmonic oscillator in 2 dimensions

The equation for the quantum mechanical oscillator in 2 dimensions is

$$\left(-\frac{1}{2}(\partial_1^2 + \partial_2^2) + \frac{1}{2}\left((x^1)^2 + (x^2)^2\right)\right) f = Ef \tag{2.47}$$

In polar coordinates it is (see exercise)

$$\left(-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2) + \frac{1}{2}r^2\right) f(r,\phi) = Ef(r,\phi) \tag{2.48}$$

With the ansatz $f(r,\phi) = f(r)e^{-i\phi\ell}$ we obtain:

$$\Big(-\frac{1}{2}(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\ell^2}{r^2}) + \frac{1}{2}r^2\Big) f(r) = Ef(r) \tag{2.49}$$

This equation has a discrete spectrum with eigenvalues

$$E_{n\ell} = 2n + \ell + 1 \tag{2.50}$$

and normalized eigenfunctions

$$f_{n\ell}(r) = e^{-r^2/2} r^\ell \sqrt{\frac{2\,n!}{(\ell+n)!}}\; L_n^\ell(r^2) \tag{2.51}$$

where L_n^ℓ are the associated Laguerre polynomials (LaguerreL$[n,\ell,r]$ in Mathematica). A few example are:

$$L_0^\ell(x) = 1; \quad L_1^\ell(x) = 1 + \ell - x; \quad L_2^\ell(x) = \frac{1}{2}\left(\ell^2 - 2x\ell + 3\ell + x^2 - 4r + 2\right) \tag{2.52}$$

Finally we can bring 2.49 into the Schrödinger-like form:

$$\left(-\partial_r^2 + \left(r^2 + \frac{4\ell^2 - 1}{r^2}\right)\right)\psi = 2E\psi \tag{2.53}$$

by putting

$$f(r) = r^{-1/2}\psi(r) \tag{2.54}$$

Therefore the solution of 2.53 is

$$\psi(r) = e^{-r^2/2} r^{\ell+1/2} L_n^\ell(r^2) \tag{2.55}$$

Experiment π	140MeV	1300 ± 100MeV	1812 ± 14MeV
$\ell = 1; \quad \lambda = 373$ MeV	745	1054	1291
$\ell = 0; \quad \lambda = 424$ MeV	850	1201	1471

Table 2.4: Experimental and soft-wall masses of the pion resonances

Results

We now come back to our soft wall potential 2.44 which we insert into eq. 2.13

We then obtain after some rearrangements the equation:

$$\left(-\frac{1}{\lambda^2}\partial_z^2 + \frac{4\ell^2 - 1}{4\,\lambda^2\,z^2} + \lambda^2 z^2\right)\psi(z) = \left(\frac{m^2}{\lambda^2} - (\kappa - 1)\right)\psi(z) \tag{2.56}$$

where we have put

$$\ell^2 = \mu^2 L^2 + \kappa + 1 \tag{2.57}$$

Introducing the new variable $u \equiv \lambda z$ we obtain

$$\left(-\partial_u^2 + \frac{4\ell^2 - 1}{4\,u^2} + u^2\right)\psi(u) = \left(m^2 - (\kappa - 1)\lambda^2\right)\psi(u) \tag{2.58}$$

By comparison with 2.53, 2.51, and 2.50 we obtain discrete values for the mass:

$$m^2 = 2E_{n\ell}\lambda^2 + 2\lambda^2 = (4n + 2\ell + 4)\lambda^2 \tag{2.59}$$

and we obtain for the solution of the original wave equation 2.41, using 2.55 and 2.46:

$$\phi(z) \sim (\lambda z)^{1/2}(\lambda z)^{\kappa/2} e^{\lambda^2 z^2}\psi(\lambda z) \sim z^{(\kappa+1)/2+\ell} L_n^\ell(\lambda^2 z^2) \tag{2.60}$$

Note that though there is no exponential decay of $\phi(z)$ it is nevertheless normalizable, since now in the scalar product the dilatation field $e^{-\lambda^2 z^2}$ is present.

The associated Laguerre Polynomials are finite for zero argument. Therefore we see that $\ell = 1$ corresponds to canonical scaling and $\ell = 0$ to scaling according to the twist dimension. The values $n = 0, 1, 2, \ldots$ give the masses of the radial exciations. As we can see, the soft wall model has the desired feature that the radial exciations are indeed proportional to the squared mass, but a direct comparison of the predicted and observed masses for the pions shows that agreement for the the soft wall model is worse than for the hard wall modell, as can be inferred from table 2.4.

The situation is different for the scalar mesons f_0 ($I = 0$, $J = 0$, $P = +$) and a_0 ($I = 1$, $J = 0$, $P = +$). Here 3 radial excitations are observed above the lowest state at 980 MeV. The experimental values and the theoretical results from the soft wall model are displayed in table 2.5. The agreement is satisfatory.

It is noteworthy that there is no good fit to the scalar mesons possible in the hard wall model, as can be seen from table 2.3. On the other hand, the high mass of the 3rd recurrence of the π ($\pi(1800)$) makes it plausible, that one resonance is missing and the $\pi(1800)$ is in fact the 4.recurrence and not the 3. one. Then the soft wall fit to the π -resonances would be much better and the soft wall model would be superior not only from a conceptual but also from a phenomenological point of view.

Experiment	f_0	980 ±10 MeV	1400 ±100 MeV	1505 ±6MeV	1724 ±7 MeV
	a_0	980 ±1.2 MeV	1474 ±19 MeV		
$\ell = 1; \lambda = 410$MeV		1004	1296	1534	1740
$\ell = 0; \lambda = 450$MeV		900	1273	1558	1800

Table 2.5: Experimental and soft-wall masses of the $f_0\,/a_0$ masses

2.1.6 Master formulæ

Here we collect some formulæ which will be used throughout the lectures. If we start with the wave equation

$$-\partial_z^2\phi - (\kappa\partial_z A - \partial_z D)\partial_z\phi + e^{2A(z)}\mu^2\phi = m^2\phi \tag{2.61}$$

with

$$A = -\log z + \log L, \qquad D = -\lambda^2 z^2. \tag{2.62}$$

We can bring 2.61 into the Schrödinger-like Form:

$$\left(-\partial_z^2 + V(z)\right)\psi = m^2\psi \tag{2.63}$$

with the potential

$$V(z) = \frac{4(\mu L)^2 + (\kappa+1)^2 - 1}{4z^2} + \lambda^4 z^2 + (\kappa - 1)\lambda^2 \tag{2.64}$$

and the wave function ϕ is related to ψ by:

$$\phi(z) = z^{\kappa/2}\, e^{\lambda^2 z^2/2}\, \psi(z) \tag{2.65}$$

For the **hard-wall model** ($\lambda = 0,\ 0 \le z \le z_0$) follows:

$$\begin{aligned}
\phi(z) &= z^{(\kappa+1)/2} J_\nu(mz) \\
\nu^2 &= (\mu L)^2 + \frac{(\kappa+1)^2}{4} \\
\nu + \frac{\kappa+1}{2} &= \ell_d; \quad \ell_d = \begin{cases} 2 \text{ for twist scaling} \\ 3 \text{ for canonical scaling} \end{cases} \\
m_{\nu,s} &= j_{\nu,s} z_0^{-1}
\end{aligned} \tag{2.66}$$

$j_{\nu,s}$ is the s'th zero of the Bessel function J_ν

For the **soft-wall model** ($\lambda \neq 0,\ 0 \le z \le \infty$) follows:

$$\begin{aligned}
\phi(z) &= z^{(\kappa+1)/2} z^\ell L_\ell^n(la^2 z^2) \\
\ell^2 &= (\mu L)^2 + \frac{(\kappa+1)^2}{4} \\
\ell + \frac{\kappa+1}{2} &= \ell_d; \quad \ell_d = \begin{cases} 2 \text{ for twist scaling} \\ 3 \text{ for canonical scaling} \end{cases} \\
m_{\nu,s}^2 &= (4n + 2\ell + (\kappa+1))\lambda^2
\end{aligned} \tag{2.67}$$

2.2 Vector particles

The Lagrangian for a vector field in AdS$_5$ is

$$\mathcal{L} = \sqrt{|g|}\frac{1}{4}(g^{MM'}\, g^{NN'} F_{MN}\, F_{M'N'}) = e^{A(z)-D(z)}\frac{1}{4}(\eta^{MM'}\, \eta^{NN'} F_{MN}\, F_{M'N'}) \tag{2.68}$$

where, as in 2.5

$$\begin{aligned}A(z) &= -\log z + \log L \qquad \text{and} \\ D(x) &= \begin{cases} 0 & \text{for the hard wall model} \\ \lambda^2 z^2 & \text{for the soft wall model}\end{cases}\end{aligned} \tag{2.69}$$

Here we started from a gauge theory in AdS_5 and have introduced no 5-dimensional mass [9, 11]. The field tensor in AdS_5 is defined analogously to the one in electrodynamics

$$F_{MN} = \partial_M A_N - \partial_N A_M \tag{2.70}$$

where we can use normal covariant derivatives because of the antisymmetric construction, see 1.40.

We proceed in a similar way as in electrodynamics (sect. 1.2.2) and obtain:

$$\frac{\partial}{\partial(\partial_K A_L)}\mathcal{L} = e^{A(z)-D(z)}\eta^{MK}\eta^{NL}(\partial_M A_N - \partial_N A_M) \tag{2.71}$$

and fom that we obtain the Euler Lagrange equation:

$$\begin{aligned}\partial_K \frac{\partial}{\partial(\partial_K A_L)}\mathcal{L} &= e^{A(z)-D(z)}\Big(\delta^5_K(\partial_z A - \partial_z D)\eta^{MK}\eta^{NL}(\partial_M A_N - \partial_N A_M) \\ &\qquad + \eta^{MK}\eta^{NL}(\partial_K\partial_M A_N - \partial_K\partial_N A_M)\Big) \\ &= e^{A(z)-D(z)}\Big((\partial_z A - \partial_z D)\eta^{M5}\eta^{NL}(\partial_M A_N - \partial_N A_M) \\ &\qquad + \eta^{MK}\eta^{NL}(\partial_K\partial_M A_N - \partial_K\partial_N A_M)\Big)\end{aligned} \tag{2.72}$$

Since we have in Minkowski space three vector particles (spin components) and 5 components of the potential, A_K, we have two gauge conditions free. We choose

$$A_5 = 0 \qquad \text{and} \quad \eta^{MK}\partial_K A_M = 0 \quad \text{(Lorenzgauge)} \tag{2.73}$$

This simplifies 2.72 to:

$$\partial_K \frac{\partial}{\partial(\partial_K A_L)}\mathcal{L} = e^{A(z)-D(z)}\Big((\partial_z A - \partial_z D)\eta^{M5}\eta^{NL}\partial_5 A_N + \eta^{MK}\eta^{NL}\partial_K\partial_M A_L\Big) \tag{2.74}$$

from which we obtain in our notation, where greek indices run from 1 to 4 and $\partial_5 \equiv \partial_z$:

$$\eta^{\nu\lambda}\Big(\eta^{\mu\kappa}\partial_\mu\partial_\kappa A_\nu - \partial_z^2 A_\nu + (\partial_z A - \partial_z D)\partial_z A_\nu\Big) = 0 \tag{2.75}$$

Now we make the ansatz (2.1):

$$A_\lambda = e^{-ipx}\epsilon_\lambda(p)\phi(z) \tag{2.76}$$

where $\epsilon(p)$ is the polarization vector of a transverse vector field, i.e. $\epsilon.p = 0$, and we obtain for $\phi(z)$ the wave equation:

$$(-\partial_z^2 - (\partial_z A - \partial_z D)\partial_z)\phi = m^2\phi \tag{2.77}$$

where $m^2 = p^2$. This is the equation 2.61 from sect. 2.1.6 with the special value $\kappa = 1$ and $\mu = 0$. We can therefore directly use the results collected there.

Exp.	$\rho(770)$ $\omega(782)$	$\rho(1465 \pm 25)$ $\omega(1420 \pm 25)$	$\rho(1720 \pm 20)$ $\omega(1670 \pm 30)$	$\rho(1900)^*$	$\rho(2150)^*$
h.w. $z_0^{-1} = 181$	692	1268	1828	2407	2976
s.w. $\lambda = 478$	956	1352	1656	1912	2138

Table 2.6: Experimental and theoretical masses of the ρ/ω resonances; all masses are in MeV; * denotes a not fully established resonance. h.w. is the hard wall model with $\nu = 1$ and $z_0^{-1} =$ 181MeV; s.w. the soft wall model with $\ell = 1$ and $\lambda = 478$ MeV

2.2.1 Hard wall model

Here we have $D(z) = 0$ and from sect. 2.1.6, 2.64 with $\lambda = \mu = 0$ we get :

$$V(z) = \frac{4-1}{4z^2}. \tag{2.78}$$

with solutions

$$\psi(z) = (zm)^{1/2} J_1(mz); \qquad \phi(z) = e^{g(z)}\psi(z) = zmJ_1(mz) \tag{2.79}$$

So the treatment with a massless vector field in 5 dimensions leads to a z^2 dependence for small z. This corresponds to the twist dimension, not the canonical dimension.

We demand that the holographic wave function vanishes at the hard wall, $\phi(z_0) = 0$. The masses m_n of the vector meson resonances are then determined by the zeros of the first Bessel function;

$$m_n z_0 = j_{1,n} \tag{2.80}$$

We now compare the hard wall model with experiment. There are 5 ρ and 3 ω resonances observed. The masses and comparison with the model is presented in table 2.6. The result for the well established resonances is neither good nor very bad.

2.2.2 Soft wall model

Here we have $\kappa = 1, \quad D(z) = \lambda^2 z^2$ and we obtain the potential:

$$V(z) = \frac{1}{2}\Big(\frac{1}{z^2} - 2\lambda^2\Big) + \frac{1}{4}\Big(-\frac{1}{z} - 2\lambda^2 z\Big)^2 = \frac{3}{4z^2} + \lambda^4 z^2 \tag{2.81}$$

with the solution

$$\psi(z) = e^{-\lambda z^2/2}(\lambda\, z)^{1/2} L_n^1(\lambda z) \qquad \phi(z) \sim (\lambda\, z)^1 L_n^1(\lambda^2 z^2) \tag{2.82}$$

and

$$m_n^2 = 2E_{n1}\lambda^2 = 4(n+1)\lambda^2 \tag{2.83}$$

Comparison with experiment, table 2.6 is more favourable for the soft wall model than for the hard wall model, especially for the higher, unfortunately not fully established resonances. It is very comforting, that the parameters for scalar and vector mesons coming from the least square fit are similar, namely $\lambda = 478$ and 450 MeV, (note that $\ell = 0$ for scalar particles gives the same scaling as $\ell = 1$ for vector particles, see 2.82 and 2.60).

It should be noted, that the reproduction of radially excited states is a notoriously difficult task in QCD. I should also mention that the determination of orbital exciations will give better results. As we shall see later, the mass of a particle with spin S will be given by:

$$m_{n,S} = 4(n+S)\lambda^2 \tag{2.84}$$

This result needs a lengthy calculation, since we cannot avoid the covariant derivatives for Spin lager than 1. We will make this calculation later.

The soft wall model is sometimes referred to as a modification of the metric. This is however excluded by the comparison of the scalar with vector mesons. If we modify the metric by introducing:

$$g_{MN} = e^{A(z)-D(z)}\eta_{MN} \tag{2.85}$$

then we would obtain as factor of the derivative terms of the Lagrangian $e^{\kappa(A(z)-D(z))}$ instead of $e^{\kappa A(z)-D(z)}$ with $\kappa = 1$ for vector mesons and $\kappa = 3$ for scalar mesons. The strength of the harmonic oscillator potential would therefore be 3 times higher for scalar then for vector mesons and correspondingly the masses multiplyed by a factor $\sqrt{3}$. We therefore have to interpret the factor $e^{-D(z)}$ as a field which influences all states equally.

2.3 Form factors

In this section we investigate the behaviour of form factors in the AdS/CFT model(following [7] and the literature quoted there). The electromagnetic form factor of a scalar particle with momentum transfer $q = p - p'$ is in AdS/CFT defined analogously to the form factor in 4-dimensions:

$$F(Q^2)(2\pi)^4\delta^4(p' - p - q)\,\epsilon.(p' - p) = \int d^4x dz\,\sqrt{|g|}g^{LL'}\Phi^*_{p'}(x,z)\,i\overleftrightarrow{\partial_L}\Phi_p(x,z)A_{L'}(x,z) \tag{2.86}$$

where Φ_p are the solutions of wave equation 2.13 discussed in section 2.1:

$$\Phi_p(x,z) = e^{-ip.x}\phi(z) \tag{2.87}$$

and $\{A_L\}$ is an external field, corresponding to the electromagnetic field in 4 dimensions with polarization vector $\{\epsilon_L\}$. This field obeys the analog of the Maxwell equations, therefore it has the same wave equation as the vector meson field (see 2.2), that is the Lagrangian is the same as given in 2.68.

We choose the gauge

$$A_5(x,z) = 0 \tag{2.88}$$

and make for the remaining four components the ansatz:

$$A_\lambda = e^{-iq.x}J(Q^2,z)\epsilon_\lambda \quad \text{where} \quad Q^2 = -q^2 \tag{2.89}$$

ϵ is here the polarization vector in 4 dimensions, with $q.\epsilon = 0$. Note that in the space like region accessible in the usual scattering experiments $Q^2 \geq 0$.

$A_\lambda(x,z)$ is the extension of an external (classical) electromagnetic potential therefore the boundary conditions will be different from the ones for the particle (quantum) fields. The formfactor is defined with respect to unit field strength, that is

$$J(Q^2,0) = 1 \tag{2.90}$$

and furthermore the total "charge", that is the field strength at $Q^2 = 0$ shall stay normalized inside the "bulk":

$$J(0,z) = 1. \tag{2.91}$$

2.3.1 Hard-wall model

The equation for $J(Q^2, z)$ is the same as for a vector meson, but $p^2 = m^2$ is replaced by $-Q^2$. In the hard wall model we had for $z \leq z_0$ no modification of the Lagrangian and had obtained the equation (see 2.77, 2.78)

$$(-\partial_z^2 + \frac{3}{4z^2})\psi(z) = m^2\,\psi(z) \tag{2.92}$$

therefore in the case considered here we get (with $J = z^{1/2}\psi$):

$$(-\partial_z^2 + \frac{3}{4z^2})\psi(z) = -Q^2\,\psi(z) \tag{2.93}$$

The solution of 2.92 was $(mz)^{1/2}J_1(mz)$, where J_1 was the regular solution of the Bessel differential equation

$$\left(-\partial_y^2 - \frac{1}{y}\partial_y + \frac{\nu^2}{y^2}\right) J_\nu(y) = J_\nu(y). \tag{2.94}$$

In order to solve 2.93, where the left hand side has a negative sign, we have to look at the modified Bessel equation

$$\Big(-\partial_y^2 - \frac{1}{y}\partial_y + \frac{\nu^2}{y^2}\Big)K_\nu(y) = -K_\nu(y); \tag{2.95}$$

the solution $K_\nu(y)$ is the modified Bessel function of order ν (BesselK[ν, z]) which is **singular** at $z = 0$.

The reason for choosing the singular solution instead of the regular are the boundary conditions 2.90, 2.91. Exactly as in the case of the vector mesons (see previoue section), we can construct the solution $J(Q^2, z)$ from the solution of a Bessel equation, but this time the modified Bessel equation 2.95, as

$$J(Q^2, z) = zQ\,K_1(Q, z) \tag{2.96}$$

(compare with 2.79) The modified Bessel function behaves for $\nu \geq 1$ as

$$K_\nu(z) = 2^{\nu-1}(\nu-1)!z^{-\nu} \tag{2.97}$$

and therefore $J(Q^2, z)$ given by 2.96 is finite at $z = 0$ and has the right normalization. Therefore 2.96 is the final result. If we had chosen the regular solution of 2.95 we had obtained a vanishing value at $z = 0$.

We obtain thus in the hard wall model:

$$\begin{aligned}\int d^4x dz\,\sqrt{|g|}g^{LL'}\,\Phi^*_{p'}(x, z)\,i\partial_L\Phi_p(x, z)A_{L'} \;&= \\ (2\pi)^4\delta^4(p' - p - q)\,\epsilon.(p' - p)\int_0^{z_0} dz\,\frac{L^3}{z^3}\,(\phi(z))^2\,(Qz)K_1(Qz)&\end{aligned} \tag{2.98}$$

where for (pseudo)scalar mesons we have

$$\Phi(z) = \frac{1}{N^2_{\nu,s}}(mz)^2 J_\nu(mz) \tag{2.99}$$

The normalization of the wave functions has to be chosen in such a way, that the form factor is equal to 1 for Q^2=0, that is:

$$N^2_{\nu,s} = \int_0^{z_0} dz\,\frac{L^3}{z^3}(mz)^4(J_\nu(mz))^2 = \frac{j^2_{\nu,s}}{z_0^2}\int_0^{j_{\nu,s}} dy\,y\,(J_\nu(y))^2 \tag{2.100}$$

where we have used that

$$m = j_{\nu,s}/z_0,$$

see 2.38, ($j_{\nu,s}$ is the s'th zero of J_ν which is larger than 0). Using the relation

$$\int_0^{y_0} dy\, y\, (J_\nu(y))^2 = \frac{1}{2} {y_0}^2 \left(J_\nu(y_0)^2 - J_{\nu-1}(y_0) J_{\nu+1}(y_0) \right) \tag{2.101}$$

we obtain

$$N_{\nu,s}^2 = -\frac{j_{\nu,s}^4 L^3}{2z_0^2} J_{\nu-1}(j_{\nu,s})\, J_{\nu+1}(j_{\nu,s}) \tag{2.102}$$

The final form of the wave function is:

$$\phi(z) = \frac{1}{N_{\nu,s}^2} (mz)^2 J_\nu(mz) \quad \text{with} \quad m = j_{\nu,s}/z_0 \tag{2.103}$$

and for the form factor:

$$\begin{aligned} F(Q^2) &= \int_0^{z_0} dz\, \frac{L^3}{z^3} \left(\phi(z)\right)^2 (Qz) K_1(Qz) \\ &= \frac{1}{N_{\nu,s}^2} \int_0^{z_0} dz\, \frac{L^3}{z^3} (mz)^4 (J_\nu(mz))^2 (Qz) K_1(Qz) \end{aligned} \tag{2.104}$$

For $\nu = 0$ we can use $J_{-1}(y) = -J_1(y)$ and therefore we have

$$N_{0,s}^2 = \frac{j_{0,s}^4 L^3}{2z_0^2} J_1(j_{0,s})^2 \tag{2.105}$$

.

It is very interesting to investigate the behaviour of the form factor for large values of Q^2. For large values of y the modified Bessel-K is exponentially falling off:

$$K_1(Qz) = \sqrt{\frac{\pi}{2Qz}}\, e^{-Qz} \left(1 + O(\frac{1}{Qz}) \right) \tag{2.106}$$

Threrefore the main contribution to the integral 2.104 comes from small values of z where

$$J_\nu(Qz) = (Qz)^\nu \frac{1}{2^\nu \nu!} + O\left((Qz)^{\nu+2} \right)$$

Therefore we obtain, after substituting $y = Qz$:

$$F(Q^2) = \frac{m^4 L^3}{N_\nu^2} \frac{1}{Q^2} \left(\frac{m}{Q} \right)^{2\nu} \int_0^{z_0 Q} dy\, y^2 \left(\frac{y^{2\nu}}{2^{2\nu} (\nu!)^2} \right) K_1(y) \left(1 + O\left(\frac{1}{Q^2} \right) \right) \tag{2.107}$$

The integral can be performed analytically and we obtain a power behaviour in Q^2:

$$F(Q^2) = \left(\frac{1}{z_0\, Q} \right)^{2+2\nu} \frac{-4(\nu+1)(j_{\nu,1})^{2\nu}}{J_{\nu-1}(j_{\nu,1})\, J_{\nu+1}(j_{\nu,1})} \left(1 + O(\frac{1}{Q^2}) \right) \tag{2.108}$$

It is remarkable that a stringlike theory gives a power behaviour for a form factor, since an extended charge distribution leads generally to an falloff faster than any power, and a power-like

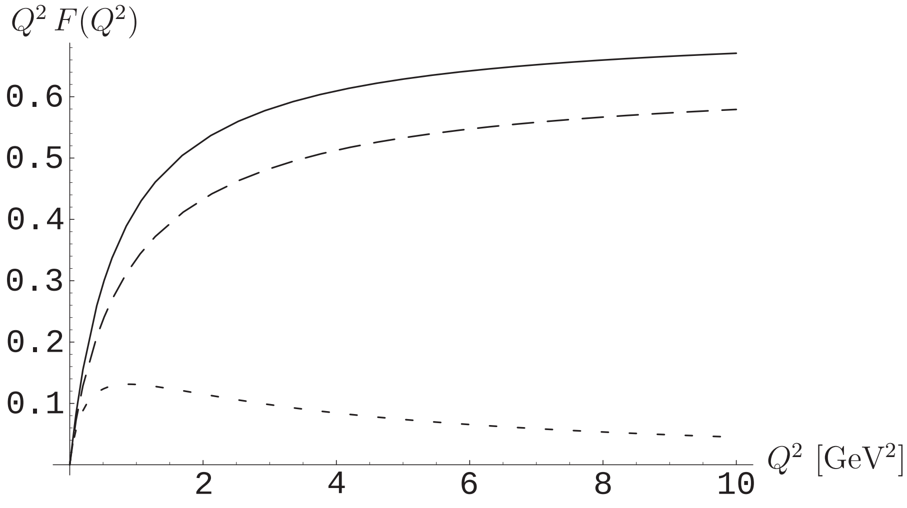

Figure 2.2: The form factor of a (pseudo)scalar particle (see sect. 2.1.4 and 2.1.5) times Q^2 in the hard wall model with $\nu = 0$; $z_0^{-1} = 207$ MeV (long dashes), $\nu = 1$; $z_0^{-1} = 167$ MeV (short dashes) and in the soft wall model with $\ell = 0$, $\lambda = 424$ MeV (solid line). The data for the π-meson agree with the models quite well (see [7], fig 4), the error bars for large Q^2 are very large, furthermore the asymptotic value of $Q^2\,F(Q^2)$ depends crucially on the input parameters, as can be inferred from 2.108

behaviour is generally related to point-like constituents. So we see that the string theory in 5 dimensions has as holographic limit in 4 dimensions indeed a theory with point-like constituents.

But there is even more physics in 2.108: As early as 1973 Brodsky and Farrar [12, 13] had postulated the scaling law that for a hadron with n constituents the form factor scales like

$$F(Q^2) \sim 1/(Q^2)^{n-1} \tag{2.109}$$

This is due to the fact that at least $(n-1)$ gluons have to be exchanged in order to distribute the momentum transfer to the constituents, and each gluon propagator contributes a factor $1/Q^2$. Since the pion has two constituents (a quark and an antiquark) the formfactor is supposed to decrease as $1/Q^2$, The scaling behaviour 2.109 together with 2.108 therefore lead to the requirement $\nu = 0$, that is scaling of the scalar operator according to the **twist dimension** ($\ell_d = 2$). The formfactors for the ground state with $\nu = 0$ (twist scaling) and $\nu = 1$ (canonical scaling are displayed in fig. 2.2 and 2.3.

2.3.2 Soft-wall model

Here we go back to the wave equation for the vector meson in the soft-wall model, 2.77, and replace m^2 by $-Q^2$, obtaining:

$$(-\partial_z^2 - (\partial_z A - \partial_z D)\partial_z + Q^2)J = 0 \tag{2.110}$$

for the soft wall model we have $(\partial_z A - \partial_z D) = -\frac{1}{z} - 2\lambda^2 z$ and hence

$$(-\partial_z^2 + (\frac{1}{z} + 2\lambda^2 z)\partial_z + Q^2)J = 0 \tag{2.111}$$

For the vector meson the solution had as variable $(\lambda^2 z^2)$, therefore we make for J the ansatz:

$$J = J(Q^2, y) \quad \text{with} \quad y = \lambda^2 z^2 \tag{2.112}$$

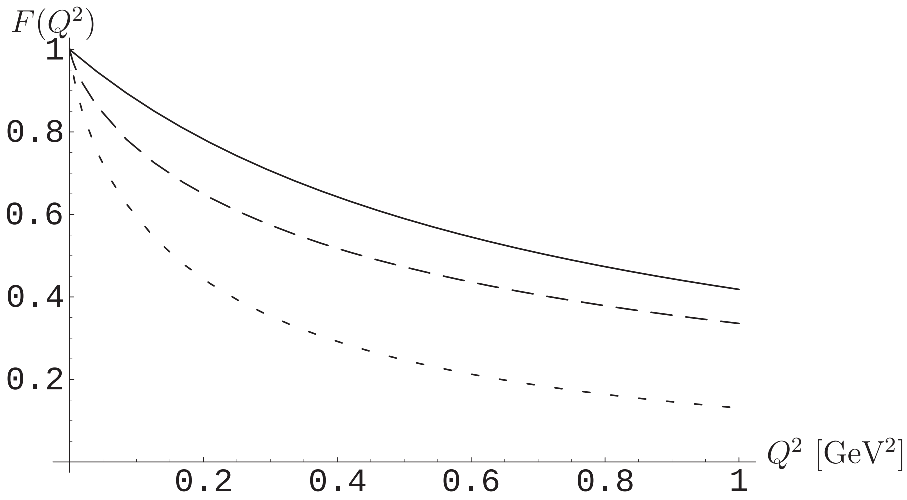

Figure 2.3: The ground state form factor of a (pseudo)scalar particle in the hard wall model with $\nu = 0$ (long dashes), $\nu = 1$ (short dashes) and in the soft wall model with $\ell = 0$ (solid line), parameters as in the previous figure.

Using

$$\partial_z J = 2\lambda^2 z \partial_y J; \qquad \partial_z^2 = 2\lambda^2 \partial_y J + 4\lambda^4 z^2 \partial_y^2 J$$

we obtain from 2.111 after dividing by $4\lambda^2$:

$$y\partial_y^2 J - y\partial_y J - \frac{Q^2}{4\lambda^2} J = 0 \tag{2.113}$$

This equation is a special case of Kummer's equation:

$$y\partial_y^2 w + (b - y)\partial_y w - aw = 0 \tag{2.114}$$

It has as a regular solution the hypergeometric function $U(a, b, y)$ (in mathematica: HypergeometricU[a,b,y]), therefore a solution of our original equation 2.111 (or equivalently 2.113) is

$$U(\frac{Q^2}{4\lambda^2}, 0, \lambda^2 z^2)$$

Some useful relations for the function U are:

$$U(\frac{Q^2}{4\lambda^2}, 0, 0) = \frac{1}{\Gamma(1 + \frac{Q^2}{4\lambda^2})} \tag{2.115}$$

$$U(0, 0, z) = 1 \tag{2.116}$$

$$\lim_{Q^2\to\infty} \Gamma(1 + \frac{Q^2}{4\lambda^2})U(\frac{Q^2}{4\lambda^2}, 0, \lambda^2 z^2) = QzK_1(Qz) \tag{2.117}$$

From this we obtain:

$$J(Q^2, z) = \Gamma(1 + \frac{Q^2}{4\lambda^2})U(\frac{Q^2}{4\lambda^2}, 0, \lambda^2 z^2) \tag{2.118}$$

is the solution with the right boundary conditions 2.90, 2.91.

Because of the asymptotic relation 2.117 the behaviour of the form factor at large values of Q^2 is the same for the soft wall model as for the hard wall model, and especially the gratifying asymptotic power behaviour in $1/Q^2$ of the form factor from the hard-wall model (see 2.108) is also valid for the form factor of the soft-wall model.

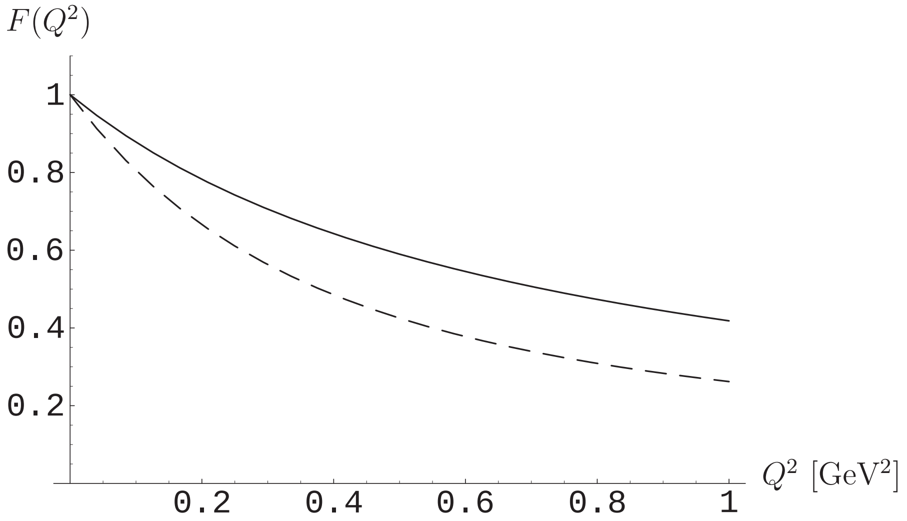

Figure 2.4: The form factor in the soft wall model for the ground state ($n = 0$, $\ell = 0$, $\lambda = 434$ MeV, solid line) and for the first radial excitation ($n = 1$, $\ell = 0$, $\lambda = 434$ MeV, dashes).

The general solution is thus:

$$F(Q^2) = \int dz \left(\frac{L^3}{z^3}\right) e^{-\lambda^2 z^2} (\phi(z))^2 \Gamma(1 + \frac{Q^2}{4\lambda^2}) U(\frac{Q^2}{4\lambda^2}, 0, \lambda^2 z^2) \tag{2.119}$$

For the soft wall model we had found the wave function of scalar mesons (see 2.60) (where $\ell = 0$):

$$\phi(z) = \frac{1}{N_{nl}} z^{2+\ell} L_n^\ell(\lambda^2 z^2) \tag{2.120}$$

The normalization condition $F(0)$=1 yields:

$$N_{n,\ell}^2 = \int dz \frac{L^3}{z^3} e^{-\lambda^2 z^2} \, z^{4+2\ell} \left(L_n^\ell(\lambda^2 z^2)\right)^2. \tag{2.121}$$

This gives as form factor for (pseudo)scalar mesons:

$$F(Q^2) = \frac{L^3}{N_{n,\ell}^2} \frac{1}{\lambda^{2+2\ell}} \int dy \, y^{2\ell+1} \, \left(L_n^\ell(y^2)\right)^2 \, e^{-y^2} \, \Gamma(1 + \frac{Q^2}{4\lambda^2}) U(\frac{Q^2}{4\lambda^2}, 0, y^2) \tag{2.122}$$

In fig. 2.4 The form factor for the ground state ($n = 0, \ell = 0$) and the first radial excitation ($n = 1, \ell = 0$) are displayed. As one expects, the excited state is more extended and therefore its form factor falls of more rapidly.

It is remarkable that in the soft wall model the expressions for the form factor 2.119 can be continued into the time like region ($Q^2 < 0$). In figure 2.6 the form factor for the ground state is displayed. One sees the pole of the ρ intermediate state, compare fig. 2.5, but not the poles corresponding to the excited states.

The form factor of the first radial excitation, displayed in fig. 2.7, shows the coupling to the radial excitations of the ρ-meson.

The model thus predicts that the excited ρ states are not coupled to the two pion state. Indeed , the $\rho(1700)$ couples dominantly to the $\rho \pi \pi$ channel and also for the $\rho(1450)$ the $\pi \pi$ channel is at least not dominant. Also in the experimental points of the form factor for the timelike region the higher ρ states are by far not so strongly present as the ground state (see [7] Fig. 5 e.g.)

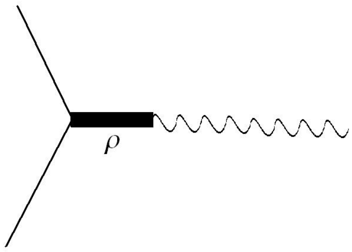

Figure 2.5: The contribution of the ρ resonances to the form factor

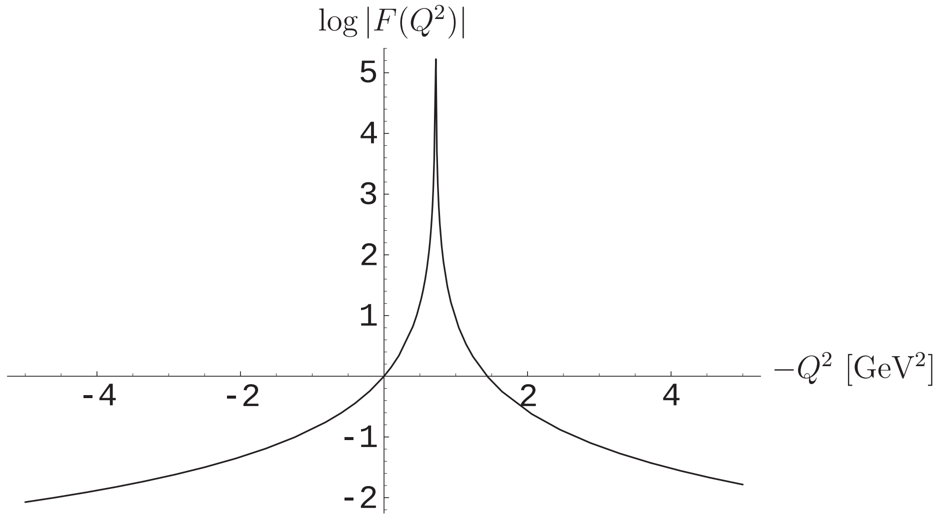

Figure 2.6: The logarithm of the form factor evaluated for the ground state in the soft wall model ($n = 0$, $\ell = 0$, $\lambda = 434$ MeV), note the pole at $2\ \lambda^2$, corresponding to the ρ pole, and the absence of poles at the ρ excited states, in order to get the ρ-pole at the right position one had to choose $\lambda = m_\rho/2$

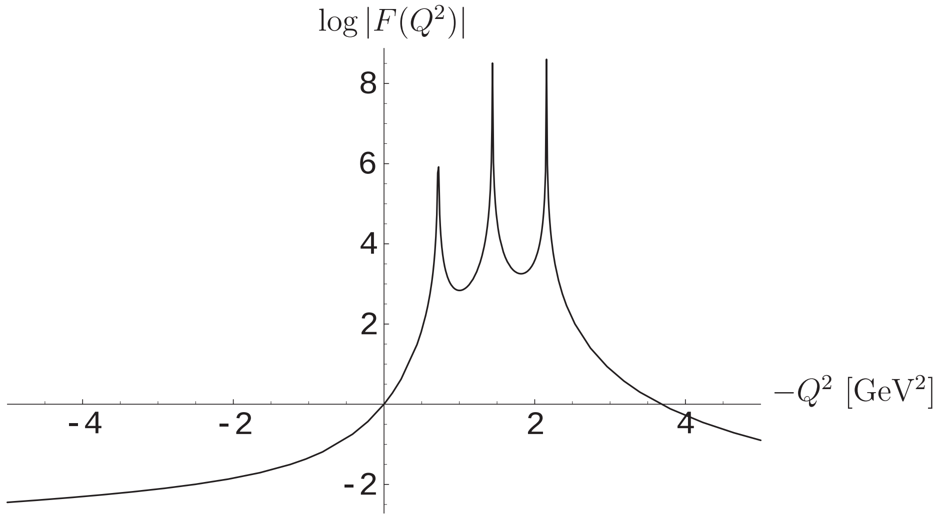

Figure 2.7: The logarithm of the form factor evaluated for the 1. excited state ($n = 1$, $\ell = 0$, $\lambda = 434$ MeV) in the soft wall model, note the poles corresponding to the ρ and the first two excited states of the ρ

$Q^2 = -4\lambda^2$					$Q^2 = -8\lambda^2$					$Q^2 = -12\lambda^2$				
n	0	1	2	3	n	0	1	2	3	n	0	1	2	3
0	-0.5	0.5	0	0	0	0	-0.5	0.5	0	0	0	0	-0.5	0.5
1	0.5	-1.5	1	0	1	-0.5	2	-3	1.5	1	0	-1	3.5	-4.5
2	0	1	-2.5	1.5	2	0.5	-3	7	-7.5	2	-.5	3.5	-11	18
3	0	0	1.5	-3.5	3	0	1.5	7.5	15	3	0.5	-4.5	18	-40

Table 2.7: The integral $K^k_{n,n'}$ (2.125) for $k = 1$ ($\rho(770)$), $k = 2$ ($\rho(1450)$), $k = 3$ ($\rho(1700)$) . One can read from this table, that in the model the 2nd excited state of the π, the $\pi(1800)$ (n=2), does not couple to the $\rho\pi$, since $K^1_{0,2} = 0$, but that the $\pi(1300)$ does, since $K^1_{0,1} = 0.5$

The residues of the poles can be obtained using the relations:

$$\begin{aligned} \Gamma(1-z) &= \frac{1}{\Gamma(z)} \frac{\pi}{\sin(\pi z)} \quad \text{and} \\ U(-k,0,z) &= (-1)^k k! L_k^{-1}(z) \end{aligned} \tag{2.123}$$

We give here the expression not only for diagonal form factors but also for transition form factors from the radial state n to n'.

$$F^k_{n,n'} = (Q^2)(q^2 - 4\lambda^2\, k) = -k\, 4\lambda^2 \frac{L^3}{N_{n,\ell} N_{n',\ell}} \frac{1}{\lambda^2} K^k_{n,n'} \tag{2.124}$$

with

$$K^k_{n,n'} = \int dy\, y\, \left(L_n^{(0)}(y^2) \right)\, e^{-y^2}\, L_k^{(-1)}(y^2) \tag{2.125}$$

The results for $K^k_{n,n'}$ for $k = 1$ ($\rho(770)$), $k = 2$ ($\rho(1450)$), $k = 3$ ($\rho(1700)$) are displayed in table 2.7.

2.4 Light cone wave functions

Much what we know about the inner structure of the hadrons we have learned from form factors. So it is not astonishing that we can also learn something about the hadron structure from the form factores which were theoretically derived expressions in the last section.

We follow the procedure of [7]. The form factor is calculated in terms of light cone wave functions (Drell-Yan West formula). The so obtained expressions are compared with the ones obtained in the previous section and from the comparison conclusions on the light cone wave functions as obtainable by AdS/CFT are drawn.

2.4.1 Light cone wave functions

We introduce the light cone coordinates:

$$x^+ = x^4 + x^3, \quad x^- = x^4 - x^3, \quad \vec{x}_\perp = \{x^1, x^2\} \tag{2.126}$$

$$p_+ = p_4 + p_3, \quad p_- = p_4 - p_3, \quad \vec{p}_\perp = \{p_1, p_2\}. \tag{2.127}$$

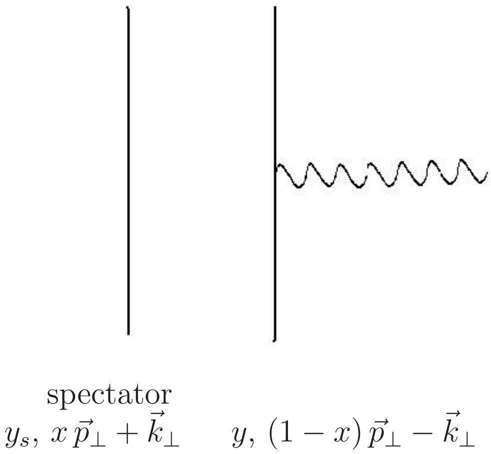

Figure 2.8: The form factor in the parton model

The light-cone wave functions are constructed by quantizing at fixed light-cone time $x^+ = 0$. Therefore the light-cone states refer to a reference frame, in which the longitudinal momentum p_3 goes to infinity. We denote a hadron with longitudinal momentum p_+ and transverse momentum $\vec{p}_\perp$ by $|p_+, \vec{p}_\perp\rangle$. For simplicity we consider first a Fock state of a hadron consisting of only two partons (a quark and an antiquark) and we suppress spin indices. Then we have:

$$|p_+, \vec{p}_\perp\rangle = \frac{1}{N}\frac{1}{(2\pi)^2}\int dx\, d^2k_\perp\, \psi(x, \vec{k}_\perp)|xp_+, x\vec{p}_+ + \vec{k}_\perp, (1-x)p_+, (1-x)\vec{p}_\perp - \vec{k}_\perp\rangle \tag{2.128}$$

where the two partons have longitudinal momentum $x\vec{p}_+$ and $(1-x)\vec{p}_+$ and transverse momenta $x\vec{p}_\perp + \vec{k}_\perp$ and $(1-x)\vec{p}_\perp - \vec{k}_\perp$, respectively, The integration over the longitudinal momentum fraction x runs always from 0 to 1.

The form factor is defined as (cf. 2.86):

$$F({q_\perp}^2) = \delta({p_+}' - p_+)\delta^2({\vec{p}_\perp}' - \vec{p}_\perp - \vec{q}_\perp)\langle p'_+, {\vec{p}_\perp}'|J(q_\perp)|p_+, \vec{p}_\perp\rangle \tag{2.129}$$

J is scalar operator multiplying the polarization vector of the external field, $A = \epsilon\, J$.

2.4.2 The form factor in light-cone formulation

We want now to calculate the form factor in the parton model, according to graph 2.8.

In position space, the two parton state is given by :

$$\begin{aligned}&\langle y_s, y|xp_+, x\vec{p}_+ + \vec{k}p, (1-x)p_+, (1-x)\vec{p}_\perp - \vec{k}_\perp\rangle = \\ &\qquad \frac{1}{(2\pi)^3}\exp[ixp_+y_s^- - i(\vec{k}_\perp + x\vec{p}_\perp)\vec{y}_{s,\perp} + i(1-x)p_+y^- - i(-\vec{k}_\perp + (1-x)\vec{p}_\perp)\vec{y}_\perp]\end{aligned} \tag{2.130}$$

The partons are supposed to be point-like particles, therefore the operator $J(\vec{q}_\perp)$ is in position space

$$|y, y_s\rangle e^{-i\vec{q}_\perp\vec{y}_\perp}\langle y, y_s| \tag{2.131}$$

and we obtain

$$F(Q^2) = \int d^3y\, d^3y_s \langle p'_+, {\vec{p}_\perp}'|y, y_s\rangle e^{-i\vec{q}_\perp\vec{y}_\perp}\langle y, y_s|p_+, \vec{p}_\perp\rangle \tag{2.132}$$

where $Q^2 = -q^2 = {\vec{q}_\perp}^2$.

First let us consider the case where the electromagnetic current is attached to parton with space coordinates y and charge e_1, y_s are the coordinates of the spectator parton.

First we collect the δ-distributions resulting from the y and y_s integration in 2.132:

$$\begin{aligned}
&(2\pi)^6\delta(xp_+ - x'p'_+)\delta\left((1-x)p_+ - (1-x)'p'_+)\right)\delta^2(-\vec{k}_\perp - x\vec{p}_\perp + \vec{k}_\perp{}' + x'\vec{p}_\perp{}') \\
&\quad \delta^2\left(\vec{k}_\perp - (1-x)\vec{p}_\perp - \vec{k}_\perp{}' + (1-x')\vec{p}_\perp{}' - \vec{q}_\perp\right) \\
= &\quad (2\pi)^6\,\delta(p_+ - p'_+)\delta^2(\vec{p}_\perp - \vec{p}_\perp{}' - \vec{q}_\perp)\,\frac{1}{p_+}\,\delta(x-x')\delta^2\left(\vec{k}_\perp{}' - \vec{k}_\perp - x(\vec{p}_\perp - \vec{p}_\perp)\right) \quad (2.133)
\end{aligned}$$

Inserting now 2.128 into 2.132 and making use of the δ-distributions we obtain finally, with $N^2 = p_+$, taking into account the contributions of both quarks:

$$F(Q^2) = (e_1 + e_2)\frac{1}{(2\pi)^2}\int dx\, d^2k_\perp\, \psi^*(x, \vec{k}_\perp - x\vec{q}_\perp)\psi(x, \vec{k}_\perp) \quad (2.134)$$

The total charge $e_1 + e_2 = 1$.

For comparison with the form factor of the previous section it is convenient to go to position space and introduce the impact parameter distribution $\tilde{\psi}(x, \vec{b}_\perp)$:

$$\psi(x, \vec{k}_\perp) = \int d^2b_\perp e^{-i\vec{b}_\perp.\vec{k}_\perp}\tilde{\psi}(x, \vec{b}_\perp) \quad (2.135)$$

and obtain from 2.134:

$$F(Q^2) = \int dx d^2b_\perp e^{-ix\,\vec{q}_\perp\vec{b}_\perp}|\tilde{\psi}(x, \vec{b}_\perp)|^2 \quad (2.136)$$

We want now to compare this expression for the form factor with the one obtained from AdS/CFT, 2.104, 2.122. We consider the case where $\tilde{\psi}(x, \vec{b}_\perp)$ depends only on $b = |\vec{b}_\perp|$, not on the angle. Then we can perform the angular distribution in 2.134 and obtain:

$$F(Q^2) = \int dx\, bdbd\theta e^{ix|\vec{q}_\perp|b\cos\theta}|\tilde{\psi}(x, b)|^2 = 2\pi\int\int dx\, bdbJ_0(xQb)|\tilde{\psi}(x, b)|^2. \quad (2.137)$$

This is the Drell-Yan West formula in position space. Here we have used

$$\int_0^{2\pi} d\theta e^{iy\cos\theta} = 2\pi J_0(y). \quad (2.138)$$

Hard-wall model

We first consider the hard wall model. Here we had found (2.104) :

$$F(Q^2) = \int_0^{z_0} dz\, \frac{L^3}{z^3}\phi(z)^2(Qz)K_1(Qz). \quad (2.139)$$

We have somehow to match the $J_0(xQb)$ in the DYW form factor 2.137 with the $QzK_1(Qz)$ from the AdS form factor 2.139. For that we use the relation:

$$Qz\, K_1(Qz) = \int_0^1 dx J_0\left(Q\, z\sqrt{\frac{x}{1-x}}\right). \quad (2.140)$$

We insert 2.140 into 2.139

$$F(Q^2) = \int_0^1 dx \int 0^{z_0} dz\, \frac{L^3}{z^3}\,(\phi(z))^2\, J_0\left(Q\, z\sqrt{\frac{x}{1-x}}\right) \quad (2.141)$$

We see that it is advantageous to substitute the integration variable z by $b = \frac{z}{\sqrt{x(1-x)}}$ obtaining

$$F(Q^2) = \int_0^1 dx \int_0^{z_0/\sqrt{x(1-x)}} db \, \frac{L^3}{x(1-x)b^3} \left(\phi(\sqrt{x(1-x)}\, b)\right)^2 J_0\,(x\,Q\,b) \tag{2.142}$$

Comparing this equation 2.142 with 2.137 we find a solution,

$$2\pi|\tilde{\psi}(x,b)|^2 = \frac{1}{x(1-x)} \frac{L^3}{b^4} |\phi(b\sqrt{x(1-x)})|^2 \tag{2.143}$$

For the scalar mesons we have found, see 2.103:

$$\phi(z) = \frac{1}{N_{\nu,s}^2} (mz)^2 J_\nu(mz) \quad \text{with} \quad m = j_{\nu,s}/z_0 \tag{2.144}$$

and therefore we have for the impact parameter distribution:

$$\sqrt{2\pi}\,|\tilde{\psi}(x,b)| = \begin{cases} \frac{m^2 L^{3/2}}{N_{\nu,s}} \sqrt{x(1-x)}\,|J_\nu(m\sqrt{x(1-x)}\,b)| & \text{for } \sqrt{x(1-x)}\,b < z_0 \\ 0 & \text{else} \end{cases} \tag{2.145}$$

The solution is, as mentioned, not unique. A requirement for the solution is that it is symmetric under interchange of x and $(1-x)$ since both partons have to be treated on the same footing.

Soft-wall model

Before we discuss further the results, we compare with he soft-wall model. There we had found:

$$F(Q^2) = \int dz \left(\frac{L^3}{z^3}\right) e^{-\lambda^2 z^2} (\phi(z))^2 \Gamma(1 + \frac{Q^2}{4\lambda^2}) U(\frac{Q^2}{4\lambda^2}, 0, \lambda^2 z^2) \tag{2.146}$$

Here we have no relation corresponding to 2.140, but we know that for large Q^2 the function $J(Q^2, z^2)$ obtained in the soft wall model agrees with that of the hard wall model (see sect. 2.3.2, esp. 2.117.) Therefore we can for large Q^2 use 2.143 also for the soft wall model.

$$2\pi\,|\tilde{\psi}(x,b)|^2 = \frac{1}{x(1-x)} \frac{L^3}{b^4} e^{-\lambda^2 b^2 x(1-x)} |\phi(b\sqrt{x(1-x)})|^2 \tag{2.147}$$

with (see 2.120 and 2.121)

$$\phi(z) = \frac{1}{N_{n\ell}} z^{2+\ell} L_n^\ell(\lambda^2 z^2); \qquad N_{n,\ell}^2 = L^3 \int dz\, e^{-\lambda^2 z^2}\, z^{2+2\ell} \left(L_n^\ell(\lambda^2 z^2)\right)^2 \tag{2.148}$$

for spinless mesons (where $\ell = 0$).

For the ground state we then obtain:

$$\sqrt{2\pi}\,\tilde{\psi}(x,b) = \sqrt{2}\lambda\sqrt{x(1-x)} e^{-\lambda^2 b^2 x(1-x)/2} \tag{2.149}$$

In fig. 2.9 we display this functions $\tilde{\psi}(x,b)$ in 3D-plots. Since the variable b is multiplied by $x(1-x)$ the light-cone wave function can extend to very large values and in the limit $x \to 0$ or $x \to 1$ they are not confined. The difference between the hard and the soft wall model becomes

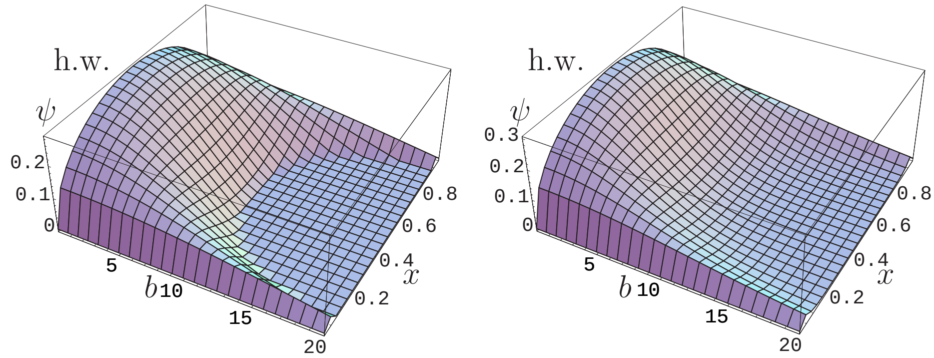

Figure 2.9: The light-cone wave function $\tilde{\psi}(x, b)$ for the hard wall (H.w.) and soft wall (s.w.) model. The transverse separation is in GeV^{-1} ($5\ \mathrm{GeV}^{-1} \approx 1$ fm).

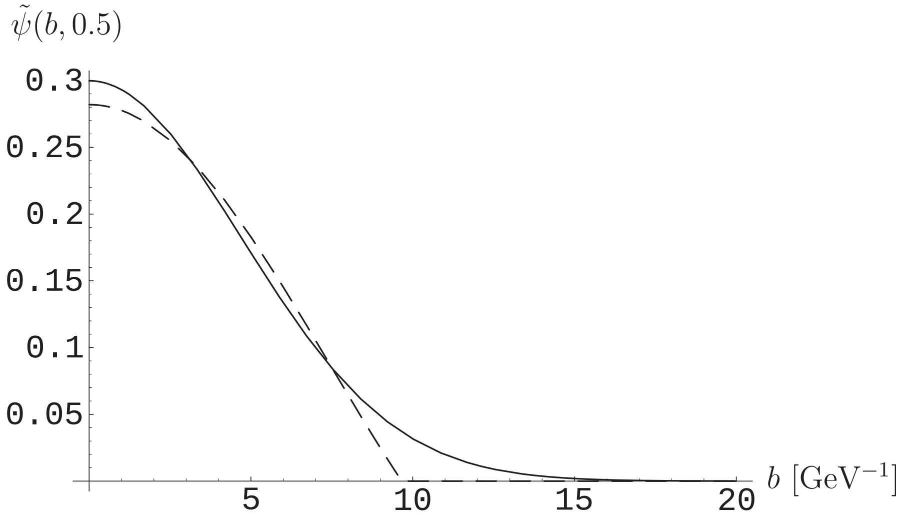

Figure 2.10: The light-cone wave functions for x=0.5, solid line: soft wall model, dashed line: hard wall model

clear in figure 2.10 where $\psi[b, \frac{1}{2}]$ is displayed for both models. As expected, the soft wall leads also to a softer decay of the wave function with increasing b.

The expression 2.147 is justified only for large values of Q^2. but Q^2 does not occur at all in eq. 2.149. Brodsky and Téramond argue, that the result of AdS with dilatational symmetry breaking modifications and the Drell Yan West form factor refer to different representations. The modified AdS/CFT result refers to a Heisenberg represenation with dressed quarks, whereas the light-cone calculations have to be done in the interaction representation and at high values of Q^2 the two representations converge.

In order to see the difference, we compare the form factor as obtained directly in AdS/CFT (sect. 2.3.2) with the result obtained by applying the light-cone wave function 2.149 to the Drell-Yan West formula 2.136. We then obtain:

$$F_{DYW}(Q^2) = \frac{1}{2\pi} 2\lambda^2 \int_0^1 dx\, x(1-x) \int d^2 b_\perp\, e^{ix\,\vec{q}_\perp.\vec{b}_\perp} e^{-\lambda^2 b^2 x(1-x)} \tag{2.150}$$

The Fourier transform of the Gaussian is easily obtained by quadratic completion:
For convenience we derive the important formula:

$$\begin{aligned} \int d^2x \exp(-\alpha^2 \vec{x}^{\,2} - i\vec{x}.\vec{k}) &= \int d^2x \exp\left(-\alpha^2 \left(\vec{x}^{\,2} - i\frac{\vec{x}.\vec{k}}{\alpha^2} - \frac{\vec{k}^{\,2}}{4\alpha^4}\right) - \frac{\vec{k}^{\,2}}{4\alpha^2}\right) \\ &= e^{\vec{k}^{\,2}/(4\alpha^2)}\, 2\pi \int dy\, y e^{-\alpha^2 y^2} = \frac{\pi}{\alpha^2}\, e^{\vec{k}^{\,2}/(4\alpha^2)} \end{aligned} \tag{2.151}$$

We obtain:

$$F_{DYW}(Q^2) = \int_0^1 dx\, e^{-\frac{x\,Q^2}{4\lambda^2(1-x)}} \tag{2.152}$$

In fig. 2.11 we display the form factor for the soft wall model 2.119 and the one of 2.152. The discrepancy seems not very large, but there is a conceptual problem. The derivative of the form factor with respect to Q^2 at $Q^2 = 0$ is proportional to the charge radius, that is a physical quantity. If we calculate this quantity from F_{DYW} we obtain:

$$-\left(\frac{\partial F_{DYW}}{\partial Q^2}\right)_{/Q^2=0} = \int_0^1 dx\, \frac{x}{4\lambda^2(1-x)} \tag{2.153}$$

which is a divergent integral!

On the other hand the asymptotic behaviour for large Q^2 is satisfactory. Due to the exponential falloff in 2.152 the main contribution for large Q^2 comes from small values of x. Introducing the variable $y = Q^2 x$ we obtain the reasonable result:

$$F(Q^2) \sim \frac{1}{Q^2} \int_0^{Q^2} dy\, e^{\frac{y}{4\lambda^2(1-y/Q^2)}} \sim \frac{4\lambda^2}{Q^2}. \tag{2.154}$$

Before discussing these problems in some detail, we shortly quote the results for Fock components with more than two partons.

More than two partons

If we have n partons in the Fock state, then we have $n-1$ independent longitudinal momentum fractions x_i, $n-1$ independent transverse momenta $\vec{k}_\perp^{(j)}$ and correspondingly $n-1$ transverse

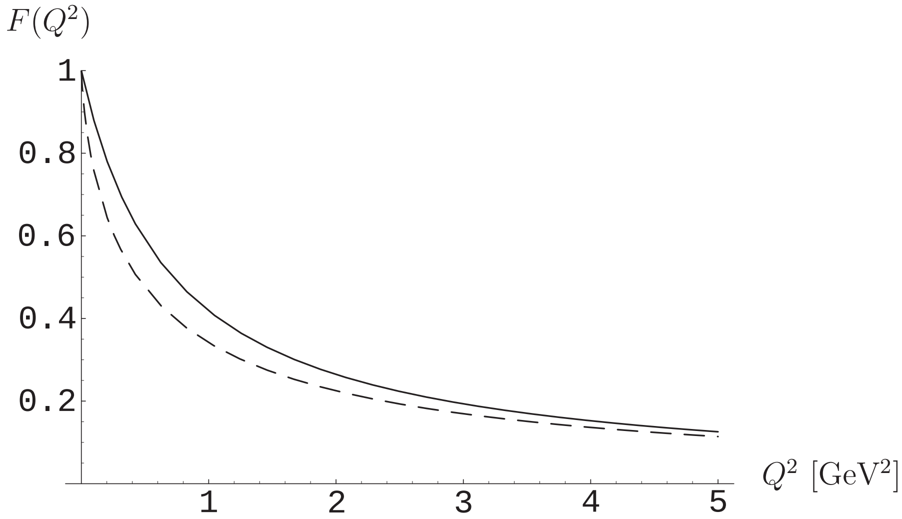

Figure 2.11: Solid line: The form factor in the soft wall model for the ground state , obtained directly from AdS/CFT. Dashed line: the form factor obtained from the soft-wall light-cone function according to eq.2.152

extensions $\vec{b}_\perp^{(j)}$ and the Drell-Yan West relation for the form factor is:

$$F(Q^2) = \sum_n \prod_{j=1}^{n-1} \int dx^{(j)}\, d^2b_\perp^{(j)} \exp\left(-i\vec{q}_\perp^{(j)}.\sum_{j=1}^{n-1} x^{(j)}\vec{b}_\perp^{(j)}\right) \left|\tilde{\psi}(x^{(j)}, \vec{b}_\perp^{(j)})\right|^2 \tag{2.155}$$

Where the $\sum_{j=1}^{n-1} x^{(j)} \leq 1$. We obtain, if there is no angular dependence and we can use 2.140:

$$2\pi \left|\tilde{\psi}(x^{(j)}, \vec{b}_\perp^{(j)})\right|^2 = x(1-x)\frac{1}{\zeta^4}|\phi(\zeta)|^2 \tag{2.156}$$

with

$$\zeta^2 = \frac{1-x}{x}\left|\sum_{j=1}^{n-1} x^{(j)}\vec{b}_\perp^{(j)}\right|^2 \tag{2.157}$$

Note that here for more than two partons the variable x has no direct meaning in the light-cone frame.

2.4.3 Extension to massive quarks

The light-cone wave functions depend on the variable $\sqrt{x(1-x)}\,b,\ b = |\vec{b}_\perp|$, a dependence which is exactly that of the Brodsky-Lepage wave functions[14]. These authors considered also light-cone wave functions for finite quark masses, it seems therefore appropriate that the procedure developped there is also applied to the wave functions considered here. For that purpose it is necessary to go back to transverse-momentum space. If we take into account that the wave function $\tilde{\psi}(x,b)$ is of the form

$$\tilde{\psi}(x,b) = \tilde{g}(x, b\sqrt{x(1-x)}\,) \tag{2.158}$$

Therefore the Fourier transform will depend on the variable $|\vec{k}_\perp|/\,(x(1-x))$:

$$\begin{aligned} \int d^2b_\perp e^{-i\vec{b}_\perp.\vec{k}_\perp}\tilde{g}(x, b\sqrt{x(1-x)}\,) &= \int d^2b'_\perp \exp\left(i\frac{|\vec{k}\,|}{\sqrt{x(1-x)}}\,b'\cos(\theta)\right)\frac{\tilde{g}(b')}{x(1-x)} \\ &= g\left(x, \frac{k_\perp^2}{x(1-x)}\right) \end{aligned} \tag{2.159}$$

where we have introduced the integration variable $b' = b\sqrt{x(1-x)}$.

The variable $\frac{k_\perp^2}{x(1-x)}$ has a physical meaning, namely it indicates the virtuality of the quarks. More precisely: The square of the total energy of a two partons with mass m_i is given by

$$\begin{aligned} E_p^2 &= (E_1+E_2)^2 = \left(\sqrt{x^2p_3^2+k_\perp^2+m_1^2} + \sqrt{(1-x)^2p_3^2+k_\perp^2+m_2^2}\right)^2 \\ &= \left(p_3 + \frac{k_\perp^2+m_1}{2xp_3} + \frac{k_\perp^2+m_1}{2(1-x)p_3} + O\left(\frac{1}{p_3^2}\right)\right)^2 \qquad (2.160) \\ &= p_3^2 + \frac{k_\perp^2+(1-x)m_1^2+xm_2^2}{x(1-x)} + O\left(\frac{1}{p_3}\right) \qquad (2.161) \end{aligned}$$

If the partons were on mass shell, this would be just the square of the energy p_4^2 of the hadron, but the latter is given by:

$$p_4^2 = p_3^2 + m_h^2 \qquad (2.162)$$

and therefore we have, up to terms of $O\left(\frac{1}{p_3^2}\right)$

$$E_p^2 - (p_4^2 - m_h^2) = \frac{k_\perp^2+(1-x)m_1^2+xm_2^2}{x(1-x)} \qquad (2.163)$$

For massless partons the right hand side of this equation is just the variable obtained above. This makes the following prescription for incorporating masses plausible: For a hadron with two quarks of mass m_1, m_2 replace the variable obtained from AdS$_5$, that is $\frac{k_\perp^2}{x(1-x)}$ by $\frac{k_\perp^2+(1-x)m_1^2+xm_2^2}{x(1-x)}$, which means replacing in 2.159

$$g\left(x, \frac{k_\perp^2}{x(1-x)}\right) \quad \text{by} \quad g\left(x, \frac{x, k_\perp^2+(1-x)m_1^2+xm_2^2}{x(1-x)}\right) \qquad (2.164)$$

The mass-modified light-cone wave function in impact space is obtained by transforming back the modified function g.

$$\tilde{\psi}(x,b) = \frac{e^{\frac{1}{\lambda^2}(m_1^2+m_2^2)}}{N(m_1,m_2)} \frac{1}{(2\pi)^2} \int d^2k_\perp e^{-i\vec{b}_\perp . \vec{k}_\perp} g\left(x, \frac{k_\perp^2+(1-x)m_1^2+xm_2^2}{x(1-x)}\right) \qquad (2.165)$$

where the factor in front of the r.h.s. must be chosen in such a way that the form factor at $Q^2=0$ is equal to 1.

For the soft wall model the result is especially simple: Using 2.151 we obtain for the Fourier transform of the wave function 2.149:

$$\psi\left(x, \vec{k}_\perp\right) = \frac{2\sqrt{\pi}}{\lambda}\sqrt{x(1-x)} \exp\left(\frac{-1}{2\lambda^2}\frac{k_\perp^2}{x(1-x)}\right) \qquad (2.166)$$

the expression $\frac{k_\perp^2}{x(1-x)}$ of the left hand side is according to our prescription replaced by

$$\frac{k_\perp^2}{x(1-x)} + \frac{(1-x)m_1^2+xm_2^2}{x(1-x)} \qquad (2.167)$$

that is the function ψ is in that case just multiplied by the factor $e^{-\frac{1}{2\lambda^2}\frac{m_1^2(1-x)+m_2^2x}{x(1-x)}}$, and we obtain, performing the Fourier transform back to impact space:

$$\tilde{\psi}(x,b,m_1,m_2) = \frac{e^{\frac{1}{\lambda^2}(m_1^2+m_2^2)}}{N(m_1,m_2)} e^{-\frac{1}{2\lambda^2}\frac{m_1^2(1-x)+m_2^2x}{x(1-x)}} \sqrt{x(1-x)} e^{-\lambda^2b^2x(1-x)/2}. \qquad (2.168)$$

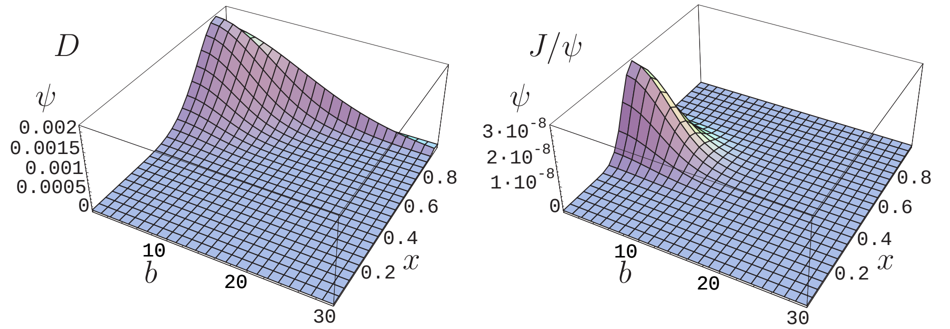

Figure 2.12: The mass modified soft wall light-cone wave function $\tilde{\psi}(x, b, m_1, m_2)$ for the D ($m_1 = 0,\ m_2 = 1.2$ GeV) and the J/ψ ($m_1 = m_2 = 1.2$ GeV) meson. The transverse separation is in GeV^{-1}.

Examples of the mass-modified functions $\tilde{\psi}(x, b, m_1, m_2)$ are displayed in figure 2.12.

It should be noted that for finite values of the mass the integral for the mean square radius 2.153 is not divergent. There is however a serious drawback in what concerns the behaviour for large Q^2. If we introduce mass terms according to 2.168, expression 2.152 is modified to

$$F(Q^2) = \frac{e^{\frac{2}{\lambda^2}(m_1^2+m_2^2)}}{N(m_1, m_2)^2} \int_0^1 dx\, e^{-\frac{x\,Q^2}{4\lambda^2(1-x)}}\, e^{-\frac{m_1^2}{\lambda^2 x} - \frac{m_2^2}{\lambda^2(1-x)}} \tag{2.169}$$

The asymptotic behaviour for large Q^2 can be evaluated analytically, using

$$\int_0^\infty dy\, e^{-a^2 y - b^2/y} = 2\frac{b}{a} K_1(2ba) \tag{2.170}$$

which leads to:

$$\begin{aligned} F(Q^2) &\sim \frac{e^{\frac{2}{\lambda^2}(m_1^2+m_2^2)}}{N(m_1, m_2)^2} \frac{1}{Q^2} \int_0^\infty dy\, e^{-y/(4\lambda^2) - m^2 Q^2/(\lambda^2 y)} \\ &\sim \frac{e^{\frac{2}{\lambda^2}(m_1^2+m_2^2)}}{N(m_1, m_2)^2} \frac{4m}{Q} K_1\left(\frac{mQ}{\lambda^2}\right) \end{aligned} \tag{2.171}$$

that is the form factor in this approach vanishes for large Q exponentially,as $\sqrt{\frac{m\lambda^2}{Q^3}}\; e^{-mQ/\lambda^2}$.

In figure 2.13 we show different results for the form factor in the soft wall model. The solid line is the direct result from AdS/CFT,(2.139), the long dashes are the result of the DYW formula with the light-cone function 2.149 and the dotted line is obtained with a mass modified light-cone wave function 2.168 with $m_1 = m_2 = \lambda/2$. We see that a finite mass improves the results for Q^2 values below ca. $3\lambda^2$, but makes the result worse for larger values of Q^2. This is another motivation for introducing Q^2 dependent masses if going from the perturbative to the non-perturbative regime. Such Q^2 dependent masses have been motivated by QCD sum rules in [15]. They are particularly important for a non-perturbative treatment of $\gamma^*\gamma^*$ interactions [16, 17].

2.5 Mesons with higher Spin

A state with higher spin S in AdS$_5$ can be expressed by a tensor field symmetric in the indices

$$\Phi_{M_1 M_2 \ldots M_S} \tag{2.172}$$

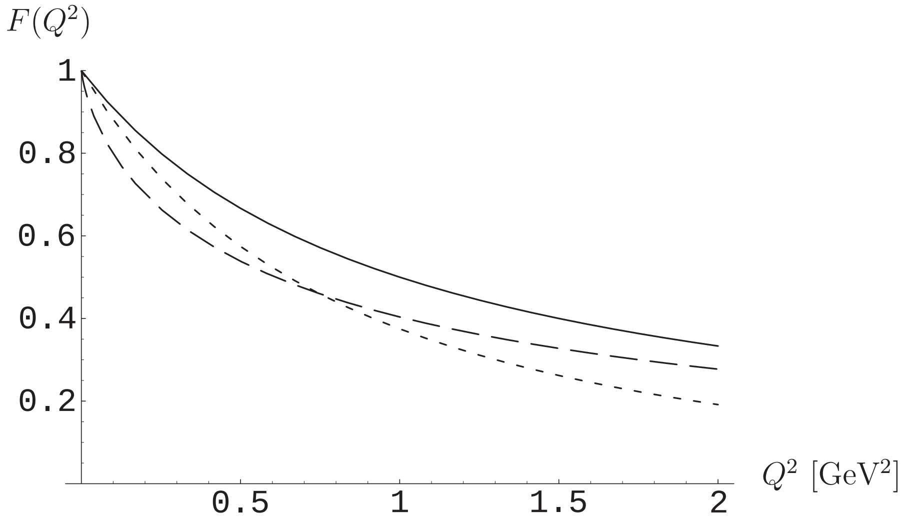

Figure 2.13: Different results for the form factor in the soft-wall model;solid line: AdS/CFT,(2.139); long dashes :DYW formula with the light-cone function 2.149; the dotted line DYW with a mass modified light-cone wave function 2.168 with $m_1 = m_2 = \lambda^2/2$

and the Lagrangian is correspondingly:

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\sqrt{||g||}\, e^{-D(z)} \Big(& g^{NN'} g^{M_1 M_1'} g^{M_2 M_2'} \dots g^{M_S M_S'} D_N \Phi_{M_1 M_2 \dots M_S}\, D_{N'} \Phi_{M_1' M_2' \dots M_S'} \\ & + \mu^2(z) g^{M_1 M_1'} g^{M_2 M_2'} \dots g^{M_S M_S'} \Phi_{M_1 M_2 \dots M_S}\, \Phi_{M_1' M_2' \dots M_S'} \\ & + \text{terms with different contractions} \Big)\end{aligned} \tag{2.173}$$

There are now two strategies possible to derive wave equations:
1) One starts in AdS$_5$ with a massless theory. Then the mass $\mu^2(z)$ term appearing above is necessary in order to cancel contributions of the covariant derivatives from the different contractions. One chooses a convenient representation for the fields in which only ordinary derivatives appear in the Lagrangian, then there is no mass term present and one can calculate the mass spectrum without further assumptions.
2) If one allows a mass term, one can start with ordinary derivatives from the beginning, since the mass term will be fixed anyhow, in our case by the scaling behaviour.

We start with the second approach, which is simpler.

2.5.1 Finite AdS$_5$ mass

We make the usual ansatz (see refmasterfield):

$$\Phi_{M_1 M_2 \dots M_S}(x, z) = e^{-ip.x} \phi(z) \epsilon_{M_1 M_2 \dots M_S} \tag{2.174}$$

where $\epsilon_{M_1 M_2 \dots M_S}$ is a symmetrical polarization tensor. We put again $A(z) = -\log z + \log L$ and $D(z) = \lambda^2 z^2$, where $\lambda = 0$ for the hard wall model. We get by the variational principle the wave equation in the same way as the one leading to 2.12:

$$e^{(3-2S)A(z)-D(z)} \left(-\partial_z^2 - \partial_z \left(\kappa A(z) - D(z) \right) \partial_z + e^{2A(z)} \mu^2 \right) \phi(z) = m^2 \phi(z) \tag{2.175}$$

with $\kappa = 3 - 2S$. We can now use again the results collected in sect. 2.1.6

Hard-wall model

Here we have $\lambda = 0$ and the boundary condition $\phi(z_0) = 0$. The solutions are:

$$\phi(z) \sim z^{\kappa/2+1/2} J_\nu(mz) = z^{2-S} J_\nu \sim z^{2-S+\nu} \tag{2.176}$$

and

$$\nu^2 = \mu^2 L^2 + (\kappa+1)^2/4 = \mu^2 L^2 + (S-2)^2 \tag{2.177}$$

The value of the mass μ has now to be determined by the scaling behaviour. The canonical dimension of the operator is $\ell_{d,c} = 3 + M$ where M is the number of derivatives in the operator. Since we can make spin 1 out of the quark spins, the minimal value of M is $S - 1$, that is the minimal value for $\ell_{d,c} = 3 + (S - 1)$ (for $S \neq 0$). The twist is defined as canonical dimension - spin, that is, we have for twist scaling always $\ell_{d,t} = 2$.

In order to get **twist** scaling we therefore must choose $\nu = S$. This is for S=1 and S=0 in agreement with our previous results, but now we have a good motivation to identify $\nu = S$. It leads to

$$\begin{aligned} m &= z_0^{-1}\zeta_{S,n} && (2.178)\\ \phi(z) &\sim z^2 \\ \mu^2 L^2 &= 4(S-1) && (2.179) \end{aligned}$$

(remember: $\zeta_{S,n}$ is the n'th zero of $J_S(y)$, n refers to the radial exciations). This is similar to the approach adopted in [7], but see the discussion at the end of this section.

If we want **canonical** scaling we must choose $\nu = 2S$ and therefore have in this case :

$$\begin{aligned} m &= z_0^{-1}\zeta_{2S,n} && (2.180)\\ \phi(z) &\sim z^{2+S} \\ \mu^2 L^2 &= 3S^2 + 4S - 4 \end{aligned}$$

Soft-wall model

Here we have

$$\phi(z) \sim z^{\kappa/2+1/2}\psi(\lambda z) \sim z^{\kappa/2+1/2+\ell} L_n^\ell(\lambda^2 z^2) \sim z^{2-S+\ell} L_n^\ell(\lambda^2 z^2) \tag{2.181}$$

and the hadron mass is given by (compare 2.59)

$$m^2 = (4n + 2\ell + 2 + (\kappa - 1))\,\lambda^2 = (4n + 2\ell + 4 - 2S)\lambda^2 \tag{2.182}$$

If we choose **twist** scaling (like z^2) we must chose $\ell = S$ and from 2.182 we obtain

$$\begin{aligned} m^2 &= 4(n+1)\lambda^2 && (2.183)\\ \phi(z) &\sim z^2 \\ \mu^2 L^2 &= 4(S-1) \end{aligned}$$

For that case the mass is independent of the spin, a certainly unreasonable result.

If we choose **canonical** scaling, then we have $2 - S + \ell = 2 + S$ and we obtain

$$\begin{aligned} m^2 &= (4n + 2S + 4)\lambda^2 && (2.184)\\ \phi(z) &\sim z^{2+S} \\ \mu^2 L^2 &= 3S^2 + 4S - 4 \end{aligned}$$

Now we get a reasonable result for the mass, but we do not get agreement with the previous case for the vector meson.

Therefore it seems very advisable to consider a massless theory in AdS$_5$, since then the scaling behaviour is fixed a priori.

2.5.2 No Mass in AdS$_5$

Here the procedure is the following[11]: A special gauge and a special representation of the fields is chosen that one has to replace the covariant derivatives in the Lagrangian by ordinary ones.

The field $\Phi_{M_1 M_2 \dots M_S}$ allows gauge transformations of the form

$$\delta\Phi_{M_1 M_2 \dots M_S} = D_{M_1}\xi_{M_2 \dots M_S} + \text{symmetrizing terms} \tag{2.185}$$

where the gauge function $\xi_{M_2 \dots M_S}$ is symmetric in the indices. We choose the gauge in which $\Phi_{5\dots} = 0$, that is if at least one of the indices of $\Phi_{M_1 M_2 \dots M_S}$ is 5 then the field is gauged to zero.

Even if the gauge is fixed in this way, we still have some gauge freedom left. We exemplify that at the example of the spin 2, that is we consider the field Φ_{M_1,M_2} and search for gauge transformation with gauge functions $\xi_5 = 0$ and which leave $\Phi_{5\mu} = 0$ invariant:

$$\delta\Phi_{5\mu} = D_5\xi_\mu + D_\mu\xi_5 = 0 \tag{2.186}$$

As we shall see immediately, it would be a fatal error to omit the second term.

We first have to obtain the covariant derivatives in AdS$_5$. Using the expressions 1.52 derived earlier

$$\begin{aligned} D_M V_N &= \partial_M V_N - \Gamma^L_{MN} V_L \\ \Gamma^L_{MN} &= \frac{1}{2} g^{LK}\left(\partial_M g_{KN} + \partial_N g_{KM} - \partial_K g_{MN}\right) \\ &= \frac{-1}{z}\left(\delta^5_M \delta^L_N + \delta^5_N \delta^L_M - \eta^{L5}\eta_{MN}\right) \end{aligned} \tag{2.187}$$

we obtain:

$$\begin{aligned} D_5\xi_\mu &= \partial_z\xi_\mu - \Gamma^\rho_{\mu\nu}\xi_\rho = \partial_z\xi_\mu + \frac{1}{z}\xi_\mu \\ D_\mu\xi_5 &= \partial_\mu\xi_5 - \Gamma^\rho_{\mu 5}\xi_\rho = \frac{1}{z}\xi_\mu \end{aligned} \tag{2.188}$$

We see that the covariant derivative of a quantity which is identically zero (ξ_5) can be different from zero. We obtain then:

$$\delta\Phi_{5\mu} = \partial_z\xi_\mu + \frac{2}{z}\xi_\mu = 0 \tag{2.189}$$

This leads to the solution:

$$\xi_\mu(x,z) = e^{2A(z)}\tilde{\xi}_\mu(x), \tag{2.190}$$

(as usual $A(z) = -\log(z) + \log(L)$).

We rescale our spin field

$$\Phi_{M_1,M_2} = e^{2A(z)}\tilde{\Phi}_{M_1,M_2} \tag{2.191}$$

then we have:

$$\begin{aligned}\delta\tilde{\Phi}_{\mu_1,\mu_2} &= e^{-2A(z)}\delta\Phi_{\mu_1,\mu_2} \\ &= e^{-2A(z)}\left(D_{\mu_1}\xi_{\mu_2}+D_{\mu_2}\xi_{\mu_1}\right) \\ &= e^{-2A(z)+2A(z)}\left(\partial_{\mu_1}\tilde{\xi}_{\mu_2}(x)+\partial_{\mu_2}\tilde{\xi}_{\mu_1}(x)\right) \end{aligned} \tag{2.192}$$

where we have used 2.190. The gauge variation of the rescaled field $\tilde{\Phi}_{\mu_1,\mu_2}$ is independent of z. The action 2.174 expressed by these rescaled fields is independent under the gauge transformations 2.190 which implies that the Lagrangian can contain only normal derivatives. We only indicate the argument schematically. The variation of the action under the gauge transformation 2.186 is

$$\begin{aligned}\delta A &= \int d^d x dz \ldots (D\,\delta\Phi).D\,\Phi + D\,\Phi.D(\delta\Phi) \\ &= \int d^d x dz \ldots (\partial_x\,(\delta\Phi).\partial_x\,\Phi + \partial_x\Phi.\partial_x(\delta\Phi) - \partial_z\,(\delta\Phi).\partial_z\,\Phi - \partial_z\Phi.\partial_z(\delta\Phi)) \qquad (2.193)\\ &\quad +\text{possibly terms from parallel transporters} \qquad (2.194)\end{aligned}$$

The ordinary part of the derivatives with respect to x, ∂_x vanishes by partial integration over x, and the differentiation with respect to z does not contribute, since because of 2.192 $\partial_z\delta\Phi = 0$. If we had also parallel transporters acting on the rescaled fields, they would yield non-vanishing contributions to the variation of the action and hence violate gauge invariance, therefore they cannot be present.

This procedure is easily extended to arbitrary $S \geq 2$ and leads to

$$\Phi_{M_1 M_2 \ldots M_S} = e^{2(S-1)A(z)}\tilde{\Phi}_{M_1 M_2 \ldots M_S} \tag{2.195}$$

The Lagrangian 2.174 in the rescaled fields is then:

$$\mathcal{L} = \frac{1}{2}\sqrt{||g||}\,e^{-D(z)}g^{NN'}g^{\mu_1\mu_1'}g^{\mu_2\mu_2'}\ldots g^{\mu_S\mu_S'}e^{4(S-1)A(z)}\partial_N\tilde{\Phi}_{\mu_1\mu_2\ldots\mu_S}\,\partial_{N'}\tilde{\Phi}_{\mu_1'\mu_2'\ldots\mu_S'} \tag{2.196}$$

collecting the coefficients of $A(z)$ we obtain:

$$\kappa = 5-2-2S+4s-4 = 2S-1 \tag{2.197}$$

which agrees with the special case of spin 1, obtained in sect. 2.2.

We now can use again the expression of 2.1.6 for the potential in the Schrödinger-like equation, but now with $\mu L = 0$ and obtain

Hard wall model

$$\nu = \frac{\kappa+1}{2} = S \quad \text{and } \tilde{\phi} \sim z^{\kappa/2+1/2}J_\nu(mz) \sim z^{2S} \tag{2.198}$$

which implies (see 2.195)

$$\phi(z) = e^{2A(z)(S-1)}\tilde{\phi}(z) \sim z^{2S-2S+2} \tag{2.199}$$

that is we get exactly the same result as for the twist scaling, 2.179 (but with $\mu L = 0$).

Soft wall model

Here we get

$$\ell = \frac{\kappa + 1}{2} = S \quad \text{and } \tilde{\phi}(z) \sim z^{\kappa/2+1/2+\ell} \sim z^{2S} \tag{2.200}$$

(see 2.195 and 2.182. This implies:

$$\begin{aligned} m^2 &= (4n^2 + 2\ell + \kappa + 1) = (4n + 4S)\lambda^2 \\ \phi(z) &\sim z^2 \end{aligned} \tag{2.201}$$

This result is very gratifying. It agrees with the result for the vector mesons (S=1) obtained previously (section 2.2.2) and has the properties expected from a string theory. The spin is proportional to the squared mass and the daughters have the same mass as the parents (see figure 1.2). This is a bonus in the bottom up approach, since it shows that the modification of the AdS$_5$ theory by the dilaton field is compatible with essential features of a string-like theory.

We finally compare the two models with experiment. In figure 2.14 the Spin of the observed ρ, ω, a, and f mesons is displayed versus the squared mass. Full symbols indicate well established resonances, open symbols not confirmed ones (according to the Particle Data Group[18]). Especially if one concentrates at the confirmed resonances, one sees clearly the structure of the linear mother trajectory and also indications of equally spaced daughter trajectories.

We compare this plot with the hard wall model ($z_0^{-1} = .189$ GeV) and the soft wall model ($\lambda = 0.5$ GeV). As can be seen from figure 2.15, the soft-wall model gives a rather decent fit to the data, whereas the hard-wall model would fail. Brodsky and Téramond argue that it is rather the angular momentum than the spin which should be identified with the ν in the hard wall model.Therefore we have plotted in figure 2.15 the predictions of the hard wall model in a plot where the angular momentum is plotted against the mass square. The agreement is satisfactory also there, but note that here the ρ has L= 0, since it has natural parity and therefore should be treated as a scalar particle, which is is in contrast to the soft wall model. Also the radial excitations are at far to high energies .

2.6 Fermions in AdS/CFT

We now treat spinors and therefore we have to find the Dirac operator in the 5 dimensional AdS metric. For that we have to solve first three problems:
1) Find Dirac-Matrices in a 5-dimensional Minkowski space
2) Find Dirac-Matrices in non-Euclidean geometry
3) Find the covariant derivative of a spinor

It will turn out that the first two problems are very easy to solve, therefore we shall treat them explicitly. For the last problem I shall only quote the main results and the solution for AdS$_5$.

Before we come to our special problems, let me make a few remarks indicating why Riemannian geometry is not sufficient to treat spin in a general way. Riemannian geometry, which we considered so far, is well suited for point mechanics. A classical particle is sufficiently described by its position and velocity. But if we move a spin along a curve, we have not only the velocity (tangent vector), but also the spin direction which may change. In a curved space, the direction of the tangent vector will change if it is moved along a closed curve, the change will be determined by the

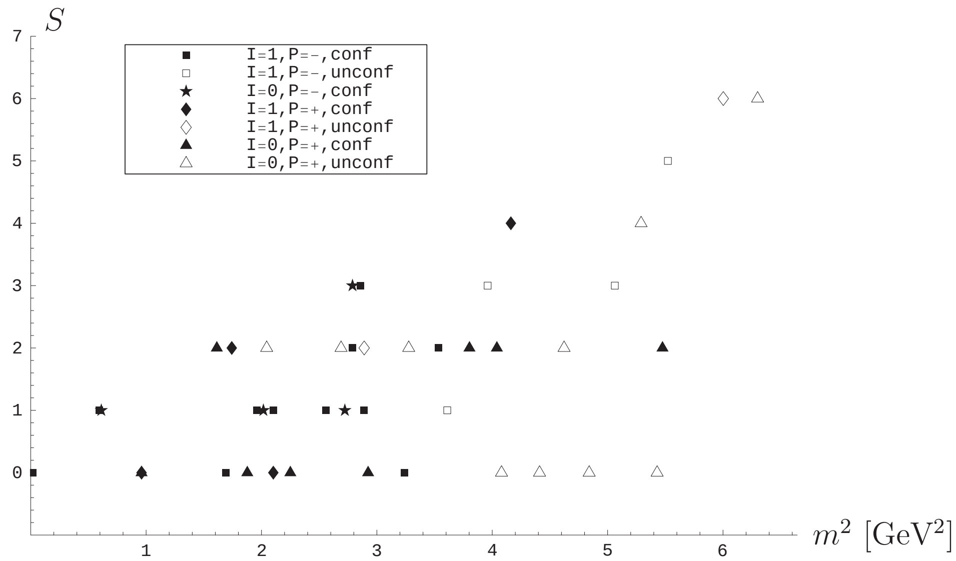

Figure 2.14: Plot of all π, ρ, ω, f, a resonances, Spin versus mass squared. The full symbols denote well confirmed resonance, the open symbols are not accepted in the PDG summary list [18]

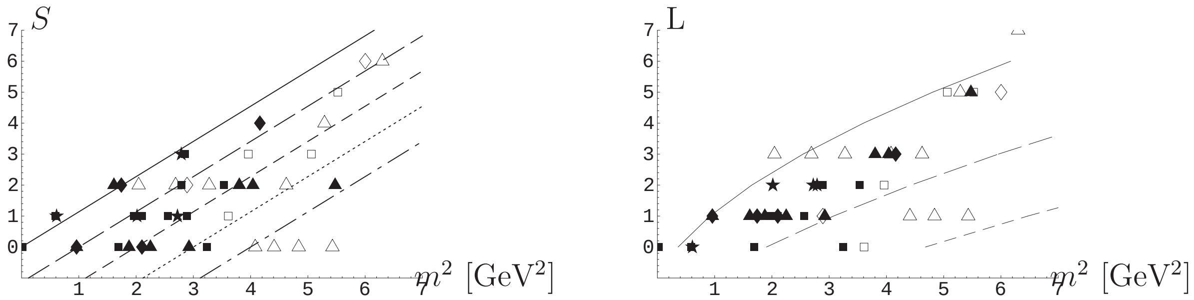

Figure 2.15: Left: experimental points as in figure 2.14, with theoretical results of the soft-wall model, $\lambda^2 = 0.22$; mother trajectory(solid), and daughters. Right:experimental points, angular momentum L vs. mass square, the lines are from the hard wall model, solid line mother trajectory (first bessel function), the daughters come from 2nd and 3rd zero., $z_0^{-}1 = 0.32$

parallel transporters. If also the spin direction changes, we say that the space has torsion, and the analogue of the parallel transporters are the spin connections. Therefore we expect that the covariant derivative, as discussed up to now, will not be sufficient for wave equations involving spin.

Parallel to the approach to non-Euclidean geometry based on the metric tensor is one based on the *'vielbein'*(many-leg) formalism. The vielbein at a point P can be seen as the coordinate system at that point for a system in which the metric tensor is just the Minkowski tensor (the so called inertial system[4]). This vielbein contains all the information contained in the metric tensor, and more: In each point there is an infinity of equivalent inertial coordinate systems, all related through Lorentz transformations. The spin connections gives us information, how the vielbein is rotated if it is moving along a curve. It turns out that the vielbein is essential for Dirac equations in non-Euclidean geometry. Especially it will allow us to solve the problem to find γ matries in non-Euclidean geometry.

2.6.1 Dirac equation in AdS$_5$

The fact that the solutions of the Dirac equation

$$(i\eta^{\alpha\beta}\gamma_\alpha\partial_\beta + m)\psi(x) = 0 \tag{2.202}$$

fulfill the energy-momentum relation of special relativity,

$$(-\eta^{\alpha\beta}\partial_\alpha\partial_\beta + m^2)\psi(x) = 0 \tag{2.203}$$

follows from the anticommutation relation:

$$[\gamma_\alpha, \gamma_\beta]_+ = 2\eta_{\alpha\beta} \tag{2.204}$$

This can be easily seen by applying the Operator $(-i\eta^{\alpha\beta}\gamma_\alpha\partial_\beta + m)$ on 2.202.

We therefore look for 5 Diracmatrices Γ_A which fulfill the relation 2.204 in a 5 dimensional Minkowski space. One can easily see that the 5 Matrices

$$\Gamma_\mu = \gamma_\mu \qquad \Gamma_5 = -i\gamma_5 \tag{2.205}$$

where γ_5 is the usual gamma matrix of the 4-dimensional Minkowski space with $(\gamma_5)^2 = 1$ and $[\gamma_5, \gamma_\mu]_+ = 0$. This solves the first problem.

The property $\gamma_5^2 = 1$, implies $\gamma_5^\dagger = \gamma_5$. Therefore $\Gamma_5^\dagger = -\Gamma_5$ or $\Gamma_0\Gamma_5^\dagger\Gamma_0 = \Gamma_5$. This is important for later calculations.

Without deeper insight we can guess immediately, that

$$\tilde{\Gamma}_A = \frac{L}{z}\Gamma_A \tag{2.206}$$

are the required matrices in AdS$_5$, since they satisfy the relation:

$$[\tilde{\Gamma}_A, \tilde{\Gamma}_B]_+ = 2\frac{L^2}{z^2}\eta_{AB}, \tag{2.207}$$

and the r.h.s. is just the AdS$_5$ metric.

But we want to derive the relation in more generality. For that we need the concept of the *vielbein*.

[4]In such a sytem in general relativity there are no forces proportional to the mass

The vielbein

We consider a d-dimensional space. As mentioned above it is always possible to find a special coordinate system in which the metric tensor is at a certain point P the Minkowski tensor. Let ξ^α be the coordinates of this point in this frame. The vielbein is a set of d vectors,

$$v_\mu^\alpha = \partial_\mu \xi^\alpha \tag{2.208}$$

where x are the coordinates of the point P in this general frame with metric tensor $g_{\mu\nu}(x)$. The following relation holds:

$$\eta_{\alpha\beta} v_\mu^\alpha v_\nu^\beta = g_{\mu\nu} \tag{2.209}$$

Proof: We consider two coordinate systems with coordinates x and x' and metric tensors $g_{\mu\nu}(x)$, $g'_{\alpha\beta}(x')$. The length of the line element is given by:

$$(ds)^2 = g'_{\alpha\beta}\, dx'^\alpha\, dx'^\beta = g'_{\alpha\beta}\, \partial_\mu x'^\alpha dx^\mu\, \partial_\nu x'^\beta dx^\nu = \left(g'_{\alpha\beta}\, \partial_\mu x'^\alpha\, \partial_\nu x'^\beta\right) dx^\mu dx^\nu \tag{2.210}$$

from which follows immediately:

$$g_{\mu\nu} = \left(g'_{\alpha\beta}\, \partial_\mu x'^\alpha\, \partial_\nu x'^\beta\right) \tag{2.211}$$

If we take as coordinates x' the inertial coordinates ξ^α which by definition have the metric tensor $\eta_{\alpha\beta}$ we get immediately 2.209.

If we define generally

$$\tilde{\gamma}_\mu = v_\mu^\alpha\, \gamma^\alpha \tag{2.212}$$

the newly defined gamma-matrices fulfill, see 2.204 and 2.209, the relation:

$$[\tilde{\gamma}_\mu \tilde{\gamma}_\nu]_+ = 2 g_{\mu\nu} \tag{2.213}$$

For AdS$_5$ the inertial coordinates are $\xi^A = \frac{1}{z} x^A$ therefore we have there

$$v_M^A = \partial_M \frac{L}{z} x^A = \frac{L}{z} \delta_M^A \tag{2.214}$$

and we see that 2.206 is indeed a special case of 2.212.

The spin connection

The covariant derivative for a spinor is given by:

$$D_\mu \psi = \left(\partial_\mu + \frac{i}{4} \omega_\mu^{\alpha\beta} \sigma_{\alpha\beta}\right)\psi \quad \text{with} \quad \sigma_{\alpha\beta} = \frac{1}{2i}[\gamma_\alpha, \gamma_\beta] \tag{2.215}$$

$\sigma_{\alpha\beta}$ is the generator for Lorentz transformations of a spinor and the spin connection indicates the behaviour of the vielbein if transported.

$$\omega_\mu^{\alpha\beta} = \frac{1}{2}\left(v_\nu^\alpha \partial_\mu g^{\nu\nu'} v_{\nu'}^\beta + v_\nu^\alpha g^{\sigma\sigma'} v_{\sigma'}^\beta \Gamma_{\sigma\mu}^\nu - (\alpha \leftrightarrow \beta)\right) \tag{2.216}$$

For AdS this simplifies considerably and the only non-vanishing contributions are

$$\omega_\mu^{5\alpha} = -\omega_\mu^{\alpha 5} = \frac{1}{z} \delta_\mu^\alpha \tag{2.217}$$

In the last equation, as usual in our AdS- notation, Greek indices run from 1 to 4.

The spin connection $\omega_\mu^{\alpha\beta}$ can be determined from the requirement that the fully covariant derivative of a vierbein vanishes:

$$D_\mu v_\nu^\alpha = \partial_\mu v_\nu^\alpha - \Gamma_{\mu\nu}^\lambda v_\lambda^\alpha + \omega_{\mu\beta}^\alpha v_\nu^\beta = 0 \tag{2.218}$$

multiplying the expression with $g^{\nu\rho} v_\rho^\gamma$ yields:

$$2\omega_\mu^{\alpha\gamma} = g^{\nu\nu'} \left(-v_{\nu'}^\gamma \partial_\mu v_\nu^\alpha + v_{\nu'}^\gamma v_\lambda^\alpha \Gamma_{\mu\nu}^\lambda \right) - (\alpha \leftrightarrow \gamma) \tag{2.219}$$

The Dirac operator in AdS$_5$

We now put pieces together. Denoting by $\Sigma_{AB} = \frac{1}{2i}[\Gamma_A, \Gamma_b]$ we obtain from 2.215 and 2.217:

$$D_M = \partial_M + \frac{i}{4z} \left(\Sigma_{5m} - \Sigma_{M5} \right) \tag{2.220}$$

from which:

$$\begin{aligned} D_\mu &= \partial_\mu - \frac{1}{2z} \Gamma_\mu \Gamma_5 \\ D_5 &= \partial_z \end{aligned} \tag{2.221}$$

From this we obtain for the Dirac operator:

$$i\, \tilde{\Gamma}^M\, D_{M'} - m_5 = \frac{i}{L} (z\gamma^\mu \partial_\mu - \frac{d}{2} \Gamma_5 - z\Gamma_5 \partial_z) - m_5 \tag{2.222}$$

where we have used: $\tilde{\Gamma}^M = z\Gamma^M, \quad \Gamma^\mu = \gamma^\mu, \quad \gamma^\mu \gamma_\mu = d(=4)$. In order to make some relations more evident and show their general nature, we shall here always write d for the dimension of the physical Minkowski space. In our case we have of course $d = 4$. The sign of the AdS mass m_5 is not fixed.

2.6.2 Action and wave equation

The action is given by (we insert the Dilaton field D, which in the hard wall model is put to zero):

$$\begin{aligned} A &= \frac{1}{2L} \int d^d x\, dz \sqrt{|g|} e^{-D(z)}\, \bar{\Psi} \Big(i(z\gamma^\mu \partial_\mu - \frac{d}{2} \Gamma_5 - z\Gamma_5 \partial_z) - m_5 L \Big) \Psi + h.c. \\ &= \frac{1}{2L} \int d^d x\, dz \sqrt{|g|} e^{-D(z)} \Bigg(\bar{\Psi} \Big(i(z\gamma^\mu \partial_\mu - \frac{d}{2} \Gamma_5 - z\Gamma_5 \partial_z) - m_5 L \Big) \Psi \\ &\quad -i\partial_\mu \Psi^* \gamma^{\mu\dagger} \gamma^0 \Psi + \frac{id}{2} \Psi^* \Gamma_5^\dagger \gamma^0 \Psi + iz\partial_z \Psi^* \Gamma_5^\dagger \gamma^0 \Psi - m_5 L \bar{\Psi} \Psi \Bigg) \\ &= \frac{1}{2L} \int d^d x\, dz \sqrt{|g|} e^{-D(z)} \times \\ &\quad \Big(iz\bar{\Psi} \gamma^\mu \partial_\mu \Psi - iz\partial_\mu \bar{\Psi} \gamma^\mu \Psi - iz\bar{\Psi} \Gamma_5 \partial_z \Psi + iz\partial_z \bar{\Psi} \Gamma_5 \Psi - 2m_5 L \bar{\Psi} \Psi \Big) \end{aligned} \tag{2.223}$$

derivatives act always only on the expression directly to their right.

We see that the term coming from the spin connection is canceled in the action density. We can perform a partial integration in x and z in order to get rid of the derivatives of $\bar{\Psi}$. This is straightforward for the x integration, for z we have to take int o account the z dependent factor $\sqrt{|g|} = e^{dA(z)-D(z)}$ and the boundary conditions.

Doing this we arrive at:

$$\begin{aligned} A &= \frac{z}{2}\int d^dx\,dz\,e^{dA(z)-D(z)}\bar{\Psi}\left(i\gamma^\mu\partial_\mu - i\Gamma_5\partial_z + \frac{i}{2}\Gamma_5\Big(\frac{-d}{z} - \partial_z D(z)\Big) - \frac{m_5L}{z}\right)\Psi \\ &\quad + \Big[z\int d^dx\,e^{dA(z)-D(z)}\bar{\Psi}\Gamma_5\Psi\Big]_0^{z_0} \end{aligned} \tag{2.224}$$

For the soft wall model $z_0 = \infty$. We vary with respect to $\bar{\Psi}$ and deduce the Dirac equation in AdS_5:

$$\left(i\gamma^\mu\partial_\mu - i\Gamma_5\partial_z + \frac{i}{2}\Gamma_5\Big(\frac{-d}{z} - \partial_z D(z)\Big) - \frac{m_5L}{z}\right)\Psi = 0 \tag{2.225}$$

We realize, that for the unmodified model $D(z) = 0$ the operator in this equation is just the Dirac operator (2.222), the spin connection term which was canceled in the action reappears in the wave equations, through the partial integration over z. This is not an accident: The spin connection is the gauge potential of the local Lorentz symmetry and as in $U(1)$ or $SU(N)$ gauge symmetry the wave equation can be obtained by replacing the normal by the covariant derivative.

We now make the ansatz corresponding to 2.1:

$$\Psi(x,z) = e^{-ip.x}u(p)\phi(z) \tag{2.226}$$

with

$$i\gamma^\mu\partial_\mu e^{-ip.x}u(p) = m_h e^{-ip.x}u(p) \tag{2.227}$$

where $u(p)$ is the Dirac spinor of a free spin 1/2 particle with momentum p in Minkowski space. In order to solve the equation we split the the field into a right ($-$) and left ($+$) handed one:

$$\Psi_\pm(x,z) = e^{-ip.x}u_\pm(p)\phi_\pm(z) \tag{2.228}$$

with $u_\pm = \frac{1}{2}(1\pm\gamma_5)u$ that is

$$\gamma_5 u_\pm = \pm u_\pm. \tag{2.229}$$

We now insert $D = -\lambda z^2$ into 2.225 and use $\Gamma_5 = -i\gamma_5$ obtaining:

$$\left(m_h - \frac{m_5L}{z}\right)(\phi_+(z)u_+ + \phi_-(z)u_-) + \left(\partial_z - \frac{d}{2z} + \lambda^2 z\right)(\phi_+(z)u_+ - \phi_-(z)u_-) = 0 \tag{2.230}$$

From which we obtain, comparing the coefficients of u_+ and u_-:

$$\begin{aligned} \left(\partial_z - \frac{d + 2\lambda^2z^2 + 2m_5L}{2z}\right)\phi_+ &= -m_h\phi_- \\ \left(\partial_z - \frac{d + 2\lambda^2z^2 - 2m_5L}{2z}\right)\phi_- &= m_h\phi_+ \end{aligned} \tag{2.231}$$

2.6.3 Spin 3/2

Before we compare with experiment I shall make some short remarks on the spin 3/2. Generally fermions with higher spin can be described by an object with mixed spinor and tensor indices[19], to wit $\psi_{\alpha,\{\mu_1\dots\mu_n\}}$, where α is a spinor index and the μ_i are vector indices, $_{\{\dots\}}$ indicates symmetrization of the indices. Spin 3/2 particles satisfy the Rarita-Schwinger equation. In the normally quoted form it is:

$$\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\partial_\rho\psi_\sigma + m\psi^\mu = 0. \tag{2.232}$$

This looks not very convenient for extension to more then 5 dimensions. In the original paper by Rarita and Schwinger[19], there is however no γ^5 but their expression suggests the alternative form:

$$\gamma^{[\mu}\gamma^{nu}\gamma^{\lambda]}\partial_\nu + m\gamma^{[\mu}\gamma^{\lambda]}\psi_\lambda = 0 \tag{2.233}$$

where $^{[\dots]}$ means anti-symmetrization of the indices. This form is indeed easily extended to non-Euclidean geometry[20], see 2.212, 2.206:

$$\left(\tilde\Gamma^{[M}\tilde\Gamma^{N}\tilde\Gamma^{L]}\partial_N + m\tilde\Gamma^{[M}\tilde\Gamma^{L]}\right)\psi_L = 0 \tag{2.234}$$

Introducing

$$\hat\psi_A = v_A^M\Psi_M \tag{2.235}$$

and choosing the gauge $\Psi_5 = 0$ it has been shown[20] that $\hat\psi_A$ satisfies indeed an equation of the form 2.225.

This can be made at least plausible from the general form of wave equations for higher-spin fermions. They are[19]:

$$(i\gamma^\rho\partial_\rho - m)\psi_{\{\mu_1\dots\mu_n\}} = 0; \quad \gamma^{\mu_1}\psi_{\{\mu_1\dots\mu_n\}} = 0 \tag{2.236}$$

It is plausible that the general solution is obtained from the Dirac equation with covariant derivative, but all tensor indices refering to the inertial frame, that is we introduce:

$$\hat\psi_{\{A_1\dots A_n\}} = v_{A_2}^{M_1}\dots v_{A_n}^{M_n}\psi_{\{A_1\dots A_n\}} \tag{2.237}$$

which satisfy the Dirac equation with the operator 2.222:

$$\left(i\,\tilde\Gamma^M\,D_{M'} - m\right)\hat\psi_{\{A_1\dots A_n\}} = 0. \tag{2.238}$$

In AdS we have $\hat\psi_{\{A_1\dots A_n\}} = z^n\psi_{\{A_1\dots A_n\}}$

2.6.4 The hard- and soft-wall model

The hard-wall model

Here we have no dilaton, that is we put $\lambda = 0$. We obtain from 2.231 the equations:

$$(-\partial_z^2 + \frac{d}{z}\partial_z + \frac{1}{4z^2}\left(4(m_5L \mp \tfrac{1}{2})^2 - (d+1)^2\right)\phi_\pm(z) = 0 \tag{2.239}$$

These equations have been discussed with different emphasis by Téramond and Brodsky[8] and Hong et al.[21]. Whereas Hon et al. consider a model in AdS including chiral symmetry breaking and assuming that the mass of the nucleon is an effect of this breaking, Téramond and Brodsky take a very pragmatic approach and proceed in line with their treatment of Meson spectra. We shall follow this latter approach.

Téramond and Brodsky started from the Dirac operator in the 10 dimensional space $AdS_5 \otimes S_5$ (see sect. 1.1.2). This amounts to an replacement of the AdS$_{d+1}$ mass m_5 by $m_5 + \Lambda_k$ where $\Lambda_k = (d+1)/2 + k$, $k = 0, 1, \ldots$ are eigenvalues of the Dirac operator on S_{d+1}.

Comparing eq. 2.239 with the wave equations of mesons, 2.175 with the equations of sect. 2.1.6, we see that we have to insert the following parameters:

$$\kappa = d; \qquad \nu_\pm^2 = (m_5 L)^2 \mp m_5 L - \frac{d^2 + 2d}{4} + \frac{(d+1)^2}{4} = (m_5 L \mp \frac{1}{2})^2 \tag{2.240}$$

Therefore we obtain the following solutions and spectra (now with $d = 4$):

$$\phi_\pm(z) = z^{(d+1)/2} J_{\nu_\pm} = z^{5/2} J_{\nu_p m} = z^{5/2} J_{|m_5 L \mp 1/2|}(m_h\, z); \qquad m_{h\pm} = j_{|m_5 L \mp 1/2|, s} z_0^{-1} \tag{2.241}$$

We shall discuss the result together with that of the soft-wall model.

Soft-wall model

We start again from 2.231 and obtain:

$$\left(-\partial_z^2 + \left(\frac{d}{z} + 2\lambda^2 z\right)\partial_z - \frac{1}{z^2}\left(\frac{d^2}{4} + \frac{d}{2} - (m_5 L)^2 \mp m_5 L\right) - \lambda^4 z^2 - (d-1)\lambda^2\right)\phi_\pm(z) = 0 \tag{2.242}$$

It is already arousing suspicion that in this equation the dilaton term has the wrong sign. We bring this equation into the Schrödinger-like form in the usual way and obtain, see sect. 2.1.6:

$$V_\pm(z) = \frac{4(m_5 L \mp \frac{1}{2})^2 - 1}{4z^2} + (d-1)\lambda^2 + \lambda^4 z^2 - \lambda^4 z^2 - (d-1)\lambda^2 \tag{2.243}$$

The usual dilaton term is canceled by the term $\lambda^4 z^2$ in 2.242. Therefore the dilaton does not lead to confinement in the spin 1/2 case!!!

On second thought this is not so astonishing. We know that fermion mass terms break chiral symmetry and the dilaton, as we have introduced it, does not break chiral symmetry, no wonder that it does not lead to a fermion mass.

Chiral symmetry breaking "dilaton"

To find some way out we go back to the Dirac equation 2.225 which we write in the form:

$$\left(i\gamma^\mu \partial_\mu - \gamma_5 \partial_z - \gamma_5 \frac{d}{2z} - \gamma_5 \lambda^2 z - \frac{m_5 L}{z}\right)\Psi = 0 \tag{2.244}$$

The linear term $\lambda^2 z$ comes with a γ_5 and therefore does not mix right and left handed spinors, this is the reason, why it does not create a discrete spectrum for the fermions. If we just omit the γ_5 in front of the linearly rising term, we have inserted a chiral symmetry breaking term and can expect to obtain a spectrum also for spin 1/2 particles.

The resulting wave equation can be obtained easily. Now the new dilaton term acts like a mass term and we have to replace only the mass term in 2.239 according to:

$$\frac{m_5 L}{z} \to \frac{m_5 L + \lambda^2 z^2}{z} \tag{2.245}$$

which leads (see sect. 2.1.6) to:

$$V(z) = \frac{4(m_5 L \mp \frac{1}{2} + \lambda^2 z^2)^2 - 1}{4z^2} = \frac{4(m_5 L \mp \frac{1}{2})^2 - 1}{4z^2} + 2(m_5 L \mp \tfrac{1}{2})\lambda^2 + \lambda^4 z^2 \tag{2.246}$$

We see that we can get confinement for both signs of λ^2, only the relative sign to m_5 matters.

As in the previous discussions of the soft wall model, we can bring the Schroedinger-like equation, by substituting $u = |\lambda| z$ and dividing through $|\lambda^2|$ into the form:

$$\left(-\partial_u^2 + \frac{4\ell_\pm^2 - 1}{4u^2} + u^2\right)\psi_\pm(u) = \left(m_h^2/|\lambda^2| - 2(m_5 L \mp \tfrac{1}{2})\tfrac{\lambda^2}{|\lambda^2|}\right)\psi_\pm(u) \tag{2.247}$$

with $\ell_\pm = |m_5 L \mp \frac{1}{2}|$.

The solutions are

$$\phi(z)_\pm \sim z^{2+1/2}\psi_\pm(|\lambda| z) \sim z^{5/2+\ell_\pm} e^{-\||la^2|z^2/2} L_n^{\ell_\pm}(|\lambda^2| z^2) \tag{2.248}$$

and the hadron mass is given by (compare 2.59)

$$m_{h\pm}^2 = \left(4n + 2\ell_\pm(1 + \mathrm{sign}(m_5 L \mp \tfrac{1}{2})\mathrm{sign}(\lambda^2)) + 2\right)|\lambda^2| \tag{2.249}$$

Here $\mathrm{sign}(x) = x/|x|$, the sign of the real quantity x. We see two special points compared to the results for bosons: 1) The exponential decrease is also present in the full holographic wave function $\phi(z)$ and 2) The AdS_5 mass m_5 and λ^2 must have the same sign in order to get a reasonable spectrum.

The sign ambiguity allows to obtain solutions which are both left and right handed. If we change the sign of m_5 and simultaneously that of λ^2, the results 2.241, 2.248 and 2.249 remain valid, only left and right handed solutions are interchanged. We therefore can stick to positive values of m_5 and λ^2. The corresponding negative values just serve to complete the solution by the missing handed part.

2.6.5 Discussion and comparison with experiment

In order to fix the values for $m_5 L$ we ave again to discuss scaling. The canonical scaling dimension of a fermion operator without derivatives is 9/2, subtracting the spins we obtain the twist dimension 3. The behaviour both for the hard- and soft-wall model, obtained above (2.241,2.248) is

$$\phi \sim z^{5/2+|m_5 L \mp 1/2|} \tag{2.250}$$

which is half integer.

We therefore assume, as in [7], the minimal scaling to be 7/2, the smallest half-integer greater than 3. This yields as minimal value $|m_5 L| = 3/2$. As mentioned we can concentrate on positive values of $m_5 L$. For the spin 3/2 the expression for $\hat{\phi}$ was the same as for the nucleon. Since $\phi = z^{-1}\hat{\phi}$ (see 2.235) we see that in order to obtain for the spin 3/2 resonances the same scaling behaviour as for the nucleons, one has to choose $m_5 L$ by one unit higher, that is $m_5 L \geq 5/2$.

Brodsky and Téramond[7, 8] argue that the angular momentum in the hadron, L, is given by

$$\mathrm{L} = m_5 L - m_5 L_{/\mathrm{minimal}}, \tag{2.251}$$

L	m_5L	ν	baryon states	mean mass	H W
0	3/2	1	$N^{1/2+}(939)$	0.94	0.94
0	5/2	2	$\Delta^{3/2+}(1232)$	1.23	1.26
1	5/2	3	$N^{1/2-} - N^{5/2-}$, 1520-1700	1.64	1.56
1	7/2	3	$\Delta^{1/2-}(1620), \Delta^{3/2-}(1700)$		
2	7/2	3	$N^{3/2+}(1720), N^{5/2+}(1680)$		
2	9/2	4	$\Delta^{1/2+} - \Delta^{7/2+}$, 1900-1950	1.92	1.86
3	9/2	5	$N^{7/2-}(2190), N^{9/2-}(2250)$	2.2	2.15
4	9/2	5	$N^{9/2+}(2220)$		
4	11/2	6	$\Delta^{11/2+}(2420)$	2.42	2.43
5	13/2	7	$N^{11/2-}(2600)$	2.6	2.71

Table 2.8: Experimental and theoretical hard wall masses for baryons according to the scheme proposed by Brodsky and Téramond, see 2.252. Displayed is L, the value of the internal angular momentum, m_5L, ν, the observed baryon states with indication of spin and parity and masses in MeV, the mean mass of all states with the same ν, and the fitted masses in the hard-wall (H W) model, the last two columns are given in GeV; the best fit parameter for the hard-wall model is: $z_0^{-1} = 0.244$ GeV and for the soft-wall model $\lambda = 0.21$ GeV.

that is

$$\begin{aligned} \nu &= m_5L - 1/2 = L + 1 \quad \text{for the } n \\ \nu &= m_5L - 1/2 = L + 2 \quad \text{for the } \Delta \end{aligned} \tag{2.252}$$

In the usual quark model the nucleon $n^{1/2+}$ and the $\Delta^{3/2+}$ have L = 0, since their internal parity is positive, the $N^{1/2-}(1535)$, $N^{3/2-}(1520)$ have L = 1, since their internal parity is negative. For one value of m_5L there are therefore 2 values of ν and possible. In ref. [7] it is assumed that the odd L values (unnatural parity) are on the $\nu = m_5 + 1/2$ branch, the even ones on the $\nu = m_5 - 1/2$ branch. In table 2.8 masses and quantum numbers nucleon- and *Delta*-resonances are displayed together with the L, m_5L, ν values and the fitted masses according to this scheme.

An important touchstone for nonperturbative methods is the ratio of the nucleon to the ρ-meson which is experimentally:

$$R = \frac{m_N^2}{m_\rho^2} \approx 3/2 \tag{2.253}$$

In the hard wall model we have to decide which way to follow. If we follow the classical assignment both the ρ and the n have the holographic wave function $\phi(z) \sim J_1(mz)$ and therefore the mass is given by $j_{1,1}z_0^{-1}$. If one assumes a universal z_0 one obtains therefore 1 for the mass ratio. If one argues that the ρ has a holographic wave function $\phi(z) \sim J_0(mz)$[5], because it has angular momentum L= 0, then one obtains $R \approx j_{1,1}^2/j_{0,1}^2 = 2.52$.

In the soft wall model the ℓ value and hence the wave function is also the same, but for vector mesons the mass was given by:
$m_M^2 = (4n + 2\ell + 2)\lambda^2$ whereas for nucleons the relation is
$m_N^2 = (4n + 4\ell + 2)\lambda^2$,
yielding the ration 1.5 ! This result is clearly in favour of the (modified) soft-wall model.

[5] in that case the ρ would scale like z

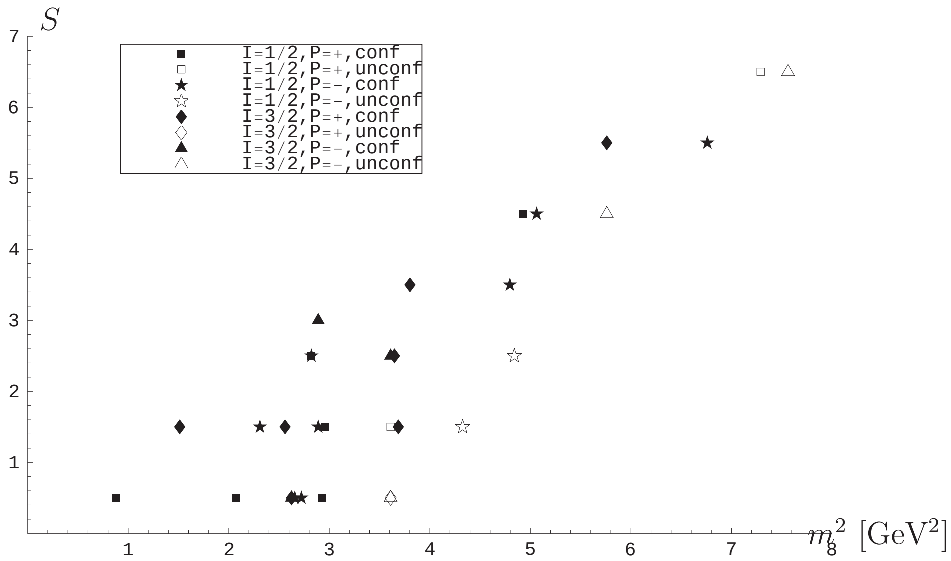

Figure 2.16: Plot of all non-strange baryon resonances, Spin versus mass squared. The full symbols denote well confirmed resonances (**** and ***), the open symbols represent ** resonances according to ref. [18]

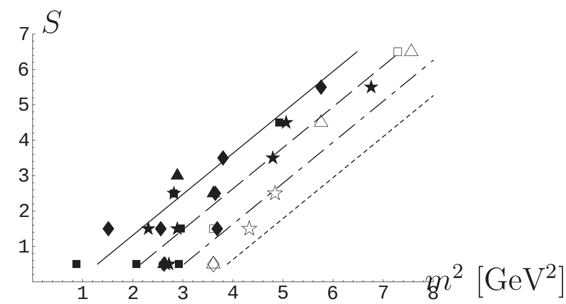

Figure 2.17: Nonstrange baryon resonances compared with theoretical predictions according to the soft wall model, with $\lambda^2 = 0.22$ GeV^2

If we plot all non-strange baryon resonances in a S vs mass-squared plot, we see again a rather distinct stringlike picture, see fig. 2.16. It is roughly in agreement with the results of the soft wall model, see fig. 2.17.

One can calculate form factors and light cone wave functions for baryons in the same way as for mesons[7].

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