QFT II - Problem Set 5

(35) Feynman Parameters

As you have seen in problem set 4, there are powerful formulae to evaluate loop terms. Sometimes, however, you need to reduce the form of these integrals to some standard form. This can be achieved due to a neat trick invented by Feynman.

a) To start, show the following idendity

$$\frac{1}{ab} = \frac{1}{b-a}\int_a^b \frac{\mathrm{d}t}{t^2}$$

b) Make a substitution $t = b + (a-b)x$, where x is the Feynman parameter and express the idendity above by an integral over z.

c) You have just shown that

$$\frac{1}{ab} = \int_0^1 \frac{\mathrm{d}x}{(xa+(1-x)b)^2} = \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\delta(x+y-1)\frac{1}{[xa+yb]^2} \tag{1}$$

d) Use this expression (1) to re-write a typical loop term

$$\frac{1}{[(k+q)^2+m^2]\,[k^2+m^2]}.$$

e) Show that

$$\frac{1}{ab^n} = \int_0^1 \mathrm{d}x\,\mathrm{d}y\,\delta(x+y-1)\frac{ny^{n-1}}{[xa+yb]^{n+1}} \tag{2}$$

d) Given the formulae (1) and (2), show by induction that

$$\frac{1}{a_1\,a_2\ldots a_n} = \int \mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n\delta\left(\sum x_i - 1\right)\frac{(n-1)!}{[x_1a_1+x_2a_2+\ldots x_na_n]^n}.$$

(36) Symmetric integrals from Minkowski to Euclidian

In exercise (34), you derived results for Euclidian space times. In order to use these for Minkowski amplitudes, you need to Wick rotate. We will need the following result for exercise (37) below. Suppose you need to integrate

$$\int \mathrm{d}^N l \frac{l^\mu l^\nu}{f(l^2)},$$

where you can take $N=4$.

a) Perform a wick rotation i.e. substitute $l^0 = -il_E^0$ for the above integral.

b) Show that

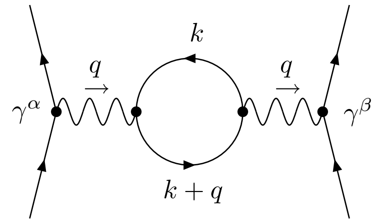
$$\int \mathrm{d}^N l \frac{l^\mu l^\nu}{f(l^2)} = \frac{-i}{N}\int \mathrm{d}^N l_E \frac{l_E^2 g^{\mu\nu}}{f(l_E^2)} \tag{3}$$

c) Show that

$$\int \mathrm{d}^N l \frac{l^2}{f(l^2)} = -i\int \mathrm{d}^N l_E \frac{l_E^2}{f(l_E^2)} \tag{4}$$

(37) VACUUM POLARISATION AND DIMENSIONAL REGULARIZATION

Consider the following process contributing toward the scattering of two electrons, depicted below.



The fermion loop in the middle surely modifies the tree level result. There will be many more graphs of this type with many more loops that all lead to the same effect: the interaction of two electrons in the full theory is different from that at tree level. Put another way: the coupling strengths, i.e. the electron charge that is physically measured will surely differ from the bare e_0 that enters the interaction Lagrangian $\mathcal{L}_{int} = e_0\bar{\Psi}A_\mu\gamma^\mu\Psi$. This effect is called *vacuum polarization* or *photon self energy*. Virtual fermion pairs shield the bare coupling e_0 and shift its value. By how much, you will compute now. The propagation from vertex α to β above can be cast in the form

$$\frac{-ig_{\alpha\beta}}{q^2} + \frac{-ig_{\alpha\mu}}{q^2}\, i\Pi^{\mu\nu}(q)\, \frac{-ig_{\nu\beta}}{q^2} + \ldots,$$

where the diagram gives the lowest order contribution, which is (verify this!)

$$i\Pi^{\mu\nu}(q) = (-ie)^2(-1)i^2 \int \frac{\mathrm{d}^N k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{\not{k}+im}{k^2+m^2}\gamma^\nu \frac{\not{k}+\not{q}+im}{(k+q)^2+m^2}\right]$$

We have written $\mathrm{d}^N k$ on purpose, because we want to evaluate this expression using dimensional regularization, i.e. we work in $N = 4-\epsilon$ dimensions.

a) Perform the trace. For this, assume that the trace formulae in $N=4$ hold in arbitrary dimensions. That's not perfectly true, but in our case sufficient. You should find

$$i\Pi^{\mu\nu}(q) = -4e^2 \int \frac{\mathrm{d}^N k}{(2\pi)^4} \frac{k^\mu(k^\nu+q^\nu) + k^\nu(k^\mu+q^\mu) - g^{\mu\nu}[k^\alpha(k_\alpha+q_\alpha)+m^2]}{[k^2+m^2][(k+q)^2+m^2]}$$

b) Use the result of (35 d) to re-write the denominator. *Hint: remember that x+y=1.*

c) Substitute $l \equiv k + xq$ and express the denominator using l.

d) Re-write the numerator in terms of l.

e) All terms linear in l in the numerator will cancel when integrated over d^4k. So what is the numerator neglecting terms linear in l ?

f) Keeping only the linear pieces of the numerator, the result is

$$Numerator = 2l^\mu l^\nu - g^{\mu\nu}l^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}[x(1-x)q^2 - m^2]$$

Move to Euclidian space and use Equation (3) and (4) to re-write the numerator as well as the denominator.

g) A useful formula you didn't compute in (34) (but you could, if you like to) is

$$\int \mathrm{d}^N l_E \frac{l_E^2}{(l_E^2+a^2)^A} = \pi^{N/2}\frac{N}{2}\frac{\Gamma(A-1-N/2)}{\Gamma(A)}\left(\frac{1}{a^2}\right)^{A-1-N/2}.$$

In addition, you showed in (34) that

$$\int \mathrm{d}^N l_E \frac{1}{(l_E^2+a^2)^A} = \pi^{N/2}\frac{\Gamma(A-N/2)}{\Gamma(A)}\left(\frac{1}{a^2}\right)^{A-N/2}.$$

So far, your result can be cast in the form

$$i\Pi^{\mu\nu}(q) = -4ie^2 \int_0^x \int \frac{\mathrm{d}^N l_E}{(2\pi)^4} \frac{I_1^{\mu\nu} + I_2^{\mu\nu} + I_3^{\mu\nu}}{(l_E^2+a^2)^2},$$

where $I_1^{\mu\nu} \equiv (1-\frac{2}{N})g^{\mu\nu}l_E^2$, $I_2^{\mu\nu} \equiv g^{\mu\nu}[x(1-x)q^2+m^2]$ and $I_3^{\mu\nu} \equiv 2x(1-x)\left[q^\mu q^\nu - q^2 g^{\mu\nu}\right]$. For each of these three contributions $I_j^{\mu\nu}$, perform the integration over $\mathrm{d}^N l_E$. *Hint: Two of them cancel nicely.*

h) Express the result of (g) in the following form

$$i\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\cdot i\Pi(q^2)$$

Convince yourself that $q_\mu \Pi^{\mu\nu} = 0$. As such, this is a result of the so called *Ward-Takahashi* idendity which is linked to gauge invariance. What's non-trivial and nice is that dimensional regularization did not destroy the idendity. A momentum cut-off would by the way not have preserved the idendity.

i) As said, we have worked in $N = 4-\epsilon$ dimensions so far. The Gamma function in $\Pi(q^2)$ diverges as $\epsilon \to 0$:

$$\Gamma[2-N/2] = \Gamma[\epsilon/2] = \frac{2}{\epsilon} - \gamma,$$

where $\gamma = 0.5772$ is the Euler-Mascheroni constant. Use this (and $x^{2-N/2} = x^{-\epsilon/2} = \exp(-\frac{\epsilon}{2}\ln x) = \dots$) to perform the limit $N \to 4$ in the expression for $\Pi(q^2)$. Retain terms of order $1/\epsilon$ and constant terms, discard all terms that vanish for $\epsilon \to 0$.

j) As said, the propagation of the photon is modified according to

$$\begin{aligned}\frac{-ig_{\alpha\beta}}{q^2} + \frac{-ig_{\alpha\mu}}{q^2}\, i\Pi^{\mu\nu}(q)\, \frac{-ig_{\nu\beta}}{q^2} + \dots \quad &= \quad \frac{-ig_{\alpha\beta}}{q^2} + \frac{-ig_{\alpha\mu}}{q^2}\, i\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)\, \frac{-ig_{\nu\beta}}{q^2} + \dots \\ &= \quad \frac{-i}{q^2(1-\Pi(q^2))}\left(g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right) + \frac{-i}{q^2}\frac{q_\alpha q_\beta}{q^2},\end{aligned}$$

where the last equal sign holds, because of the terms hidden in the "..." in the equation above. It's common to absorb this modification at $q^2 = 0$ in a redefinition of the electric charge e in terms of the bare coupling e_0 which we used to call e

$$e = \sqrt{Z_3}e_0$$

where

$$Z_3 = \frac{1}{1-\Pi(0)}.$$

At some different momentum q^2, the amplitude for the process will then involve

$$\frac{-ig_{\alpha\beta}}{q^2}\left(\frac{e_0^2}{1-\Pi(q^2)}\right),$$

instead of the simple tree-level propagator $-ig_{\alpha\beta}/q^2$.

(i) Replace e_0 by e and Π according to the formulae above.

(ii) Compute the running (q^2-dependence) of the electric charge, or put another way, what is

$$\Pi(q^2) - \Pi(0)$$