# QFT I - Problem Set 10

**(19)** Lie Algebras

We consider the exponential representation $e^{i\alpha_a X_a}$ of an unitary group. The generators $X_a$ are hermitian operators and form a vector space.
In general

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a+\beta_b)X_a} \,,$$

but as the elements form a group, it must hold

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

for some $\delta$ (summation over repeated indices is understood).

a) Show by expansion up to quadratic order in $\alpha$ and $\beta$

$$i\delta_a X_a = \ln\left(1 + e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1\right) = i\alpha_a X_a + i\beta_b X_b - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + \dots \,.$$

b) Show that the generators fulfill the following Lie algebra

$$[X_a, X_b] = i f_{abc} X_c \,.$$

The $f_{abc}$ are called structure constants and summarize the entire group multiplication law. Show that $f_{abc} = -f_{bac}$ and that the $f_{abc}$ are real (for a unitary representation).

*Remark: Expanding beyond second order in a) yields no additional conditions to make sure that the group multiplication law is maintained.*

c) As a simple and explicit example work out rotations in ordinary space ($R^3$). You can write rotations around an axis of rotation $\mathbf{u}, (|\mathbf{u}| = 1)$ by the angle $\epsilon$ as

$$R(\mathbf{u}, \epsilon) = e^{-i\epsilon \mathbf{u}\cdot\mathbf{J}} \,,$$

where the $J_1, J_2, J_3$ are the generators of the rotations around the $x-, y-, z-$axis.
Show that these generators suffice the following Lie algebra

$$[J_i, J_j] = i\epsilon_{ijk} J_k \,.$$

*Hint: One way is to directly construct adequate J's and to verify the algebra. But one can also deduce the algebra by studying two consecutive infinitesimal rotations in different order.*

**(20)** Poincaré transformations

Poincaré transformations are linear transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \text{or} \quad x' = \Lambda x + a \,,$$

with the multiplication law

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2) \,.$$

a) The unit element is $(\mathbf{1}, 0)$. What is the inverse $(\Lambda, a)^{-1}$?

The Lie algebra of the Poincaré group is generated by $4(4+1)/2$ (four space-time dimensions) generators: the $4(4-1)/2$ generators $M_{\mu\nu} = -M_{\nu\mu}$ of the Lorentz group and the 4 generators of the group of translations,

$$(\Lambda, a) = \exp\left\{\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu} + i a^\mu P_\mu\right\} \,,$$

where the variables $\omega^{\mu\nu}$ and $a^\mu$ parameterize the transformation. To get the full Poincaré algebra we need commutation relations of the infinitesimal Lorentz transformations and translations.

b) Observe that

$$(\Lambda, 0)(\mathbf{1}, a)(\Lambda, 0)^{-1} = (\mathbf{1}, \Lambda a)\,.$$

Study this relation for infinitesimal translations and show

$$[P_\rho, M_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu)\,,$$

with $\eta = \mathrm{diag}(-1, 1, 1, 1)$.

*Hint:* $(M_{\mu\nu})_{\rho\sigma} = -i(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\mu\sigma})$.

To summarize, the Poincaré algebra is (in this convention):

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}) \\
[P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
[P_\mu, P_\nu] &= 0\,.
\end{aligned}$$

## (21) Dirac Matrices

Here, we would like to exercise a bit with gamma matrices. In the convention of the lecture the $\gamma$-matrices are

$$\gamma^k = \begin{pmatrix} 0 & -i\tau_k \\ i\tau_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad \text{and} \quad \gamma^0 = \begin{pmatrix} 0 & -i\mathbf{1} \\ -i\mathbf{1} & 0 \end{pmatrix},$$

where the $\tau_k$, $k = 1, 2, 3$ are the Pauli matrices.

a) Show that the $\gamma$-matrices fulfill the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

and that $(\gamma^i)^2 = 1$ for $i = 1, 2, 3$ and $(\gamma^0)^2 = -1$.

As the square of a gamma matrix is hence $\pm 1$, the largest (“fundamental”) product of gamma matrices is the important

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

b) Show that

$$\{\gamma^\mu, \gamma^5\} = 0\,, \quad (\gamma^5)^2 = 1\,.$$

and

$$[\gamma^5, \sigma^{\mu\nu}] = 0\,.$$

with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.